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Journal of Pure and Applied Algebra 204 (2006) 280–299

JOURNAL OF PURE AND APPLIED ALGEBRA

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On the cyclic Deligne conjecture

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Received 27 October 2004; received in revised form 22 March 2005
Available online 25 May 2005
Communicated by C.A. Weibel

Abstract

Let A be a finite dimensional, unital, and associative algebra which is endowed with a non-degenerate and invariant inner product. We give an explicit description of an action of cyclic Sullivan chord diagrams on the normalized Hochschild cochain complex of A. As a corollary, the Hochschild cohomology of A becomes a Frobenius algebra which is endowed with a compatible BV operator. If A is also commutative, then the discussion extends to an action of general Sullivan chord diagrams. Some implications of this are discussed.

MSC: 16E40; 16W99; 13D03

1. Introduction

This paper is concerned with algebraic structures on the normalized Hochschild cochains, mirroring those of String Topology. String Topology may be regarded as the study of the algebraic topology of the free loop space of a manifold. Chas and Sullivan [1,2], showed that the cohomology of the free loop space of a manifold has the structure of a BV algebra. Building on Sullivan's later work [13], Cohen and Godin [3] showed that string topology

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doi:10.1016/j.jpaa.2005.04.009

operations give rise to a two dimensional positive boundary TQFT. These were achieved by looking at certain operations coming from what are known as Sullivan chord diagrams.

The Hochschild cochain complex of the singular cochains on a (simply connected) manifold gives a model for the chains on the free loop space of that manifold. One expects analogs of the above structures in a purely algebraic setting. Moreover, one is interested in algebraic structures not only at the level of Hochschild cohomology, but also, and more importantly so, at the level of Hochschild cochains. The Deligne conjecture, which has been proved in [5,7,10,14–16], partly addresses this issue. It states that the chains on the little disc operad act on the Hochschild cochain complex of an associative algebra. One relevant question is whether the chains on the framed little disc operad, or equivalently chains on cacti with marked points, act on the Hochschild cochain complex of a unital and associative algebra which has an invariant non-degenerate inner product. This question has been affirmatively answered by McClure and Smith and by Kaufmann; see [6,9,11]. The aim of the present paper is to show that a much larger set of operations, with a richer internal algebraic structure, act.

We give an explicit action of cyclic Sullivan chord diagrams (see Section 2 for definition), which include the chains on the cacti with marked points, on the normalized Hochschild cochain complex. More precisely, we show the following:

Theorem 3.3. *Let A be a finite dimensional, unital, and associative algebra with a non-degenerate and invariant inner product. Then, the normalized Hochschild cochain complex of A is an algebra over the PROP, $C_*\mathscr{S}^c$, of cyclic Sullivan chord diagrams.*

Corollary 3.4. *Under the above assumptions, the Hochschild cohomology of A is a Frobenius algebra endowed with a compatible BV operator.*

Cyclic Sullivan chord diagrams do not account for the operation which reverses the orientation of a loop. As it turns out, the concept of a Sullivan chord diagram is precisely the generalization needed for labelling the orientation reversing operations, in addition to the operations which are labelled by the cyclic Sullivan chord diagrams. In Section 4 we show that this larger PROP still acts on the normalized Hochschild complex, if the associative algebra A happens to be in addition commutative.

Theorem 4.3. *Let A be a finite dimensional, unital, associative, and commutative algebra endowed with a non-degenerate and invariant inner product. Then, the normalized Hochschild cochain complex of A is an algebra over the PROP, $C_*\mathscr{S}$, of Sullivan chord diagrams.*

Corollary 4.4. *Under the assumptions of Theorem 4.3, the Hochschild cohomology of A is a Frobenius algebra, which is endowed with a compatible BV operator, Δ, and an involution $\sim$. The operator Δ maps each eigenspace of $\sim$ into the other, i.e. $\Delta(HH^*(A;A)_\pm) \subset HH^*(A;A)_\mp$, where $HH^*(A;A)_\pm$ are the ± 1 eigenspaces of $\sim$. The map $\sim$ is both an anti-algebra and an anti-coalgebra map. That is to say $\widetilde{f \smile g} = \widetilde{g} \smile \widetilde{f}$, and $\vee_0(\widetilde{f}) = \sum_{(f)} \widetilde{f}'' \otimes \widetilde{f}'$, where $\vee_0(f) = \sum_{(f)} f' \otimes f''$.*

2. Cyclic Sullivan chord diagrams

In this section, we introduce a special kind of Sullivan chord diagrams, called cyclic Sullivan chord diagrams. For them, one defines a boundary operator, a composition, and an operation which corresponds to a certain relabelling. These diagrams yield a PROP, which is called the PROP of cyclic Sullivan chord diagrams. We will discuss how the Frobenius PROP, as well as the *BV* operad, sit inside the homology of this PROP.

A *cyclic Sullivan chord diagram* consists of a finite collection of disjointly embedded planar circles which may be connected using a finite number of immersed planar trees. Such a tree is called a chord. An endpoint of a chord lies on a circle. Different endpoints may lie on the same circle, and even on the same point. However, there may exist circles to which no chords are attached. The chords are not allowed to enter the circles. A chord has two types of vertices, the inner vertices and the endpoints, where it meets with the circles. The circles and the chords together form a graph (with possibly a collection of disjoint circles). At a vertex of this graph, there is a natural cyclic ordering of the edges, which is induced by the orientation of the plane. The cyclic ordering of the edges at each vertex gives rise to a well-defined thickening of the diagram to an oriented surface with boundary. A diagram of type $(g; n, m)$ is one for which this surface is of genus g, and has $n + m$ boundary components, precisely n of which are inside the original circles. As part of the structure, these boundaries, which are referred to as the inputs, are enumerated. Each input circle is decorated with a marked point, called the input marked point, and is oriented in a clockwise fashion. The remaining m boundary components, which are known as the outputs, are also enumerated and decorated with output marked points. We reserve the term *special point* for collectively referring to the input and output marked points, as well as to the chord endpoints.

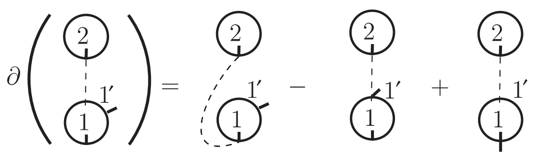
The thickened surface is merely an auxiliary tool for better picturing the input and output circles. Mathematically, all that matters is the combinatorial structure of the cyclic Sullivan chord diagram. In fact, the input and output marked points are all physically placed on the original circles. The chords are to be thought of as objects of length zero. Consequently, an output marked point or a chord endpoint which is located at an endpoint of a chord, may slide from that endpoint to an adjacent one along the perimeter of the output circle.

The output circles are oriented as follows. The induced orientation of the surface from the plane induces an orientation on its boundary components. This induced orientation, which opposes the orientation of the input circles, should coincide with the orientation of the output circles. Note that since the circles are oriented and the chords do not enter the circles, at a vertex on a circle, there is a natural linear ordering on the set of chord endpoints union the set of output marked points at that point. We would like to emphasize that in this linear ordering output marked points may very well be positioned in between chord endpoints, and are considered as part of the linear ordering.

The combinatorial dimension of a cyclic Sullivan chord diagram of type $(g; n, m)$ is by definition the number of connected components obtained by removing the special points (chord endpoints, input and output marked points) from the input circles, minus n. We only consider the cyclic Sullivan chord diagrams up to abstract combinatorial isomorphism of graphs, sending input circles to input circles, mapping chords to chords, respecting special points, and cyclic orderings at the vertices. Consequently, the orientations of the input and output circles should match as well.

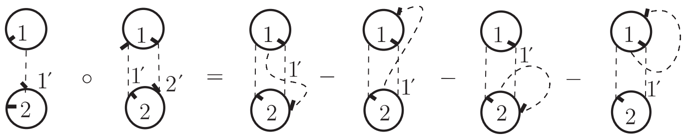
Definition 2.1 (*Cyclic Sullivan chord diagrams*). Let $C_k\mathscr{S}^c(g; n, m)$ denote the vector space generated by the cyclic Sullivan chord diagrams of type $(g; n, m)$ and of combinatorial dimension k, up to isomorphism. Let $C_k\mathscr{S}^c(n, m) = \bigoplus_{g=0}^{\infty} C_k\mathscr{S}^c(g; n, m)$, and $C_*\mathscr{S}^c(n, m) = \bigoplus_{k=0}^{\infty} C_k\mathscr{S}^c(n, m)$.

There is a natural boundary operator, ∂, on $C_*\mathscr{S}^c(n, m)$. By linearity, it suffices to describe ∂ on the basis elements. Consider a basis element $s \in C_k\mathscr{S}^c(g; n, m)$. By removing the special points (chord endpoints, the input, and the output marked points) from the input circles of s, one obtains $k + n$ connected components. Since these oriented circles are enumerated, there is a natural numbering of these connected components from 1 to $n+k$. Let $\partial(s) \in C_{k-1}\mathscr{S}^c(g; n, m)$ denote the alternating sum of all cyclic chord diagrams obtained by one at a time collapsing of each of the connected components to a point. Let us emphasize that in this paper, in defining the boundary of a diagram, we collapse no chords or any segments thereof. It is easy to verify that $\partial^2 = 0$. See the following example of the boundary of a diagram.



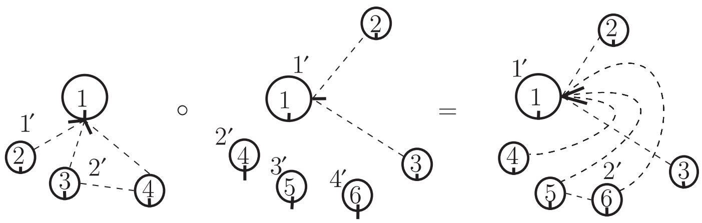
Here the input circles are labelled by 1 and 2, and the output circle is labelled by $1'$. Throughout this paper, we label the input circles using numbers $1, 2, 3, \ldots$, and the output circles with $1', 2', 3', \ldots$. For the inputs, these numbers are written inside the input circles. In case of an output, these numbers are written somewhere close to the diagram along the perimeters of the appropriate output circles. In order to better see the output circles one may, merely as a device, slightly thicken the cyclic Sullivan chord diagram to obtain an auxiliary orientable surface with boundary.

There is also a naturally defined composition. By linearity, it suffices to define the composition $\circ : C_*\mathscr{S}^c(k, l) \otimes C_*\mathscr{S}^c(m, k) \to C_*\mathscr{S}^c(m, l)$ on the basis elements. For two such elements $s \in C_*\mathscr{S}^c(k, l)$ and $s' \in C_*\mathscr{S}^c(m, k)$, we want to define $s \circ s'$. For every $1 \leqslant i \leqslant k$ consider the ith output of s' and the ith input of s. Each of these is a circle with a certain number of vertices and a particular marked point on it. Starting from the marked point of the ith input circle of s, one can read off the linear ordering of chords and output marked points arranged around this input circle. After aligning the input of s with that of s', this linear ordering should be shuffled in between the previously existing chords of the ith output circle of s' in all possible ways. The ith output circle of s' and the ith input circle of s are now dissolved. With respect to the total ordering of vertices on chord diagrams, for each chord endpoint or marked point of s which moves past an output marked point or chord endpoint of s', a sign factor of (-1) accrues. Summing over all possibilities gives rise to the desired composition. See the following example:

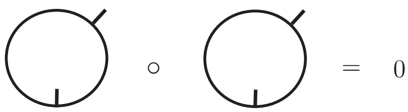


The input circles are labelled by 1 and 2, and the output circles by $1'$ and $2'$.

In the following composition example we consider a situation in which two output points and three chord endpoints of s coincide with one of its input points. In the process of identifying the corresponding circles, the marked point of the input circle of s, labelled by 1, is to be identified with the marked point of the output circle of s', labelled by 1'. At this point, it is important to keep track of the combinatorics of the chords and output points of s, together with those of s'. The chords and the output points of s should be inserted all together at one place in between those of s', respecting the linear ordering.



If the combinatorial dimension of the composed object, $s \circ s'$, is less than the sum of those of s and s', then the composition is zero. This is precisely the situation in the following composition.



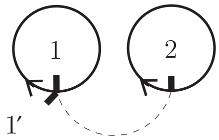
One can check that the above differential is a derivation of the composition, i.e. the composition is a chain map between the corresponding chain complexes.

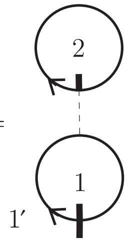
Relabelling naturally gives rise to a map from the permutation group S_n to $C_0\mathscr{S}^c(0, n, n) \subset C_0\mathscr{S}^c(n, n)$, which induces an S_n-action on suitable chord diagrams. More precisely, such a permutation corresponds to n disjoint circles in the plane, without chords, whose inputs and outputs are numbered as prescribed by the permutation. The following proposition organizes all of the above structures into a single mathematical object.

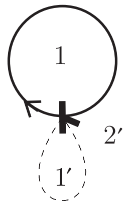
Proposition 2.2 (*PROP of cyclic Sullivan chord diagrams*). *The collection $C_*\mathscr{S}^c$, of chain complexes $C_*\mathscr{S}^c(m, n)$, for $m, n \geqslant 1$, together with the above composition rule is a PROP in the category of chain complexes. The tensor product is the disjoint union.*

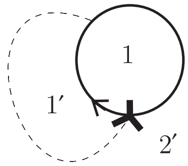
This PROP is referred to as the *PROP of cyclic Sullivan chord diagrams*.

Example 2.3. Let’s look at the following four chord diagrams.

$\smile :=$ 

and $* :=$ 

$\vee_0 :=$ 

and $\vee :=$ 

Note that $\smile \in C_0\mathscr{S}^c(2, 1)$ and $* \in C_1\mathscr{S}^c(2, 1)$. In each case the input circles are labelled by 1 and 2, and the output circle by $1'$. Similarly, $\vee_0 \in C_0\mathscr{S}^c(1, 2)$ and $\vee \in C_1\mathscr{S}^*(1, 2)$. Each input is labelled by 1, and the output circles are labelled by $1'$ and $2'$. The notation $\smile$ was taken from [4], and $\vee_0$ and $\vee$ are borrowed from [12].

Observe that $\smile$ and $\vee_0$ are both closed elements. It is straightforward to check that these elements satisfy associativity and coassociativity,

$$\smile \circ(\smile \otimes id) = \smile \circ(id \otimes \smile)$$

and

$$(id \otimes \vee_0) \circ \vee_0 = (\vee_0 \otimes id) \circ \vee_0.$$

Here $id \in C_0\mathscr{S}^c(1, 1)$ is the cyclic Sullivan chord diagram consisting of one circle without chords, whose input and output marked points coincide.

Let $\tau_2 \in C_0\mathscr{S}^c(2, 2)$ denote the element that switches the labelling, as defined by the map $S_2 \to C_0\mathscr{S}^c(2, 2)$. It is easy to see that

$$\partial(*) = \smile - (\smile \circ \tau_2)$$

and

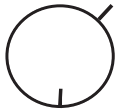
$$\partial(\vee) = \vee_0 - (\tau_2 \circ \vee_0).$$

This implies that after passing to homology, $\smile$ and $\vee_0$ are commutative and cocommutative, respectively. Finally, a word about the Frobenius compatibility conditions. We have,

$$\vee_0 \circ \smile = (id \otimes \smile) \circ (\vee_0 \otimes id) = \tau_2 \circ (\smile \otimes id) \circ (id \otimes (\tau_2 \circ \vee_0))$$

Since $\vee_0$ is cocommutative on homology, this equation implies the Frobenius compatibility condition at the level of homology.

If we define

$\Delta :=$ 

then $\Delta^2 = 0$ (see p. 5), and one may ascertain that the *BV* relation is satisfied on homology; see [1, Section 5].

$$\begin{aligned}&\Delta\circ \smile \circ(\smile \otimes id) \simeq \smile \circ(\Delta\circ \smile \otimes id) + \smile \circ(id \otimes \Delta\circ \smile)\\&\quad + \smile \circ(\Delta\circ \smile \otimes id) \circ (id \otimes \tau_2) + \smile \circ(\smile \otimes id) \circ (\Delta \otimes id \otimes id)\\&\quad + \smile \circ(\smile \otimes id) \circ (id \otimes \Delta \otimes id) + \smile \circ(\smile \otimes id) \circ (id \otimes id \otimes \Delta).\end{aligned}$$

Similarly, the dual *coBV* relations are satisfied. That is to say,

$$\begin{aligned}&(\vee_0 \otimes id) \circ \vee_0 \circ \Delta \simeq (\vee_0 \circ \Delta \otimes id) \circ \vee_0 + (id \otimes \vee_0 \circ \Delta) \circ \vee_0\\&\quad + (id \otimes \tau_2) \circ (\vee_0 \circ \Delta \otimes id) \circ \vee_0 + (\Delta \otimes id \otimes id) \circ (\vee_0 \otimes id) \circ \vee_0\\&\quad + (id \otimes \Delta \otimes id) \circ (\vee_0 \otimes id) \circ \vee_0 + (id \otimes id \otimes \Delta) \circ (\vee_0 \otimes id) \circ \vee_0.\end{aligned}$$

This, however, turns out to be true for more trivial reasons. Each individual term of the above equation is in fact homologous to zero.

Comment 2.4. Vector spaces generated by diagrams of type $(0; n, 1)$ whose chords in the plane do not cross are closely related to the operad of cacti. In a cyclic Sullivan chord diagram, collapsing each chord to a point gives rise to a cactus. In fact, the operation of collapsing chords establishes an isomorphism of operads.

3. The associative case

Now that the PROP of cyclic Sullivan chord diagrams is built, we want to make it act. Let us first recall a few relevant notations and definitions.

Let $(A, \cdot, 1)$ be a finite dimensional, unital, and associative algebra over a ground field k. A and A^* are both examples of A-bimodules. More precisely, the left and right multiplications give A an A-bimodule structure. The A-bimodule structure of $A^* := Hom(A, k)$ is given by $(a_1.a^*.a_2)(a_3) := a^*(a_2 \cdot a_3 \cdot a_1)$, for any $a_1, a_2, a_3 \in A$ and $a^* \in A^*$. Let $\beta : A \to A^*$ be an isomorphism of A-bimodules whose inverse we denote by $\gamma : A^* \to A$. Define an inner product $\langle -, - \rangle : A \otimes A \to k$ by $\langle a_1, a_2 \rangle := (\beta(a_1))(a_2)$. It is easy to verify that β is an A-bimodule isomorphism, if and only if, $\langle -, - \rangle$ is a non-degenerate bilinear map, satisfying

$$\langle a \cdot b, c \rangle = \langle a, b \cdot c \rangle$$
$$\langle a \cdot b, c \rangle = \langle b, c \cdot a \rangle$$

This implies that the map $(a_1, \ldots, a_r) \mapsto \langle a_1 \cdot \ldots \cdot a_r, 1 \rangle$ is invariant under a cyclic rotation of $a_1, \ldots, a_r$, i.e.

$$\langle a_1 \cdot \ldots \cdot a_r, 1\rangle = \langle a_r \cdot a_1 \cdot \ldots \cdot a_{r-1}, 1\rangle. \tag{1}$$

Let us recall the definition of the normalized Hochschild cochain complex and that of the endomorphism PROP.

Definition 3.1 (*Normalized Hochschild cochain complex*). Let M be a A-bimodule. The *Hochschild cochain complex of A with values in M* is the graded vector space $HC^*(A; M) := \prod_{n\geqslant 0} Hom(A^{\otimes n}, M)$, endowed with the differential

$$\begin{aligned}(\delta(f))(a_1, \ldots, a_n) := {} & a_1.f(a_2, \ldots, a_n)\\ & + \sum_{j=1}^{n-1} (-1)^j \cdot f(a_1, \ldots, a_j \cdot a_{j+1}, \ldots, a_n)\\ & + (-1)^n \cdot f(a_1, \ldots, a_{n-1}).a_n,\end{aligned}$$

where "." denotes the left and right module structures. A straightforward check shows that $\delta^2 = 0$; see e.g. [8, 1.5.1].
The *normalized Hochschild cochain complex of A with values in M* is the subcomplex

$$\overline{HC^*}(A; M) := \{f \in HC^*(A; M) | f(a_1, \ldots, a_n) = 0 \text{ if one of the } a_j = 1\}$$

It is a well-known fact that the inclusion $\overline{HC^*}(A; M) \hookrightarrow HC^*(A; M)$ is a quasi-isomorphism; see e.g. [8, 1.5.7].

The bimodule isomorphism $\beta : A \xrightarrow{\cong} A^*$ induces an isomorphism of chain complexes $\beta_\sharp : \overline{HC^*}(A; A) \xrightarrow{\cong} \overline{HC^*}(A; A^*)$, where $\beta_\sharp(f) := \beta \circ f$.

Definition 3.2 (*Endomorphism PROP*). Let V be a differential graded vector space over k. The *endomorphism PROP of V* is collection of differential graded vector spaces $\mathcal{E}nd_V(k, l) := Hom(V^{\otimes k}, V^{\otimes l})$. The map $S_k \to \mathcal{E}nd_V(k, k)$ is given by $\sigma(v_1 \otimes \cdots \otimes v_k) := (-1)^{|\sigma|} v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(k)}$, and the composition $\circ : \mathcal{E}nd_V(k, l) \otimes \mathcal{E}nd_V(m, k) \to \mathcal{E}nd_V(m, l)$ is defined by

$$(F \circ G)(v_1, \ldots, v_m) := F(G(v_1, \ldots, v_m)). \tag{2}$$

Theorem 3.3. *Let A be a finite dimensional, unital, and associative algebra with a non-degenerate and invariant inner product. Then, the normalized Hochschild cochain complex of A is an algebra over the PROP, $C_*\mathcal{S}^c$, of cyclic Sullivan chord diagrams.*

Corollary 3.4. *Under the above assumptions, the Hochschild cohomology of A is a Frobenius algebra endowed with a compatible BV operator.*

Proof of Theorem 3.3. The objective is to establish a map $\alpha : C_*\mathcal{S}^c \to \mathcal{E}nd_{\overline{HC^*}(A;A)}$ which respects the differentials, composition, and symmetric group action. We will achieve this in four steps.

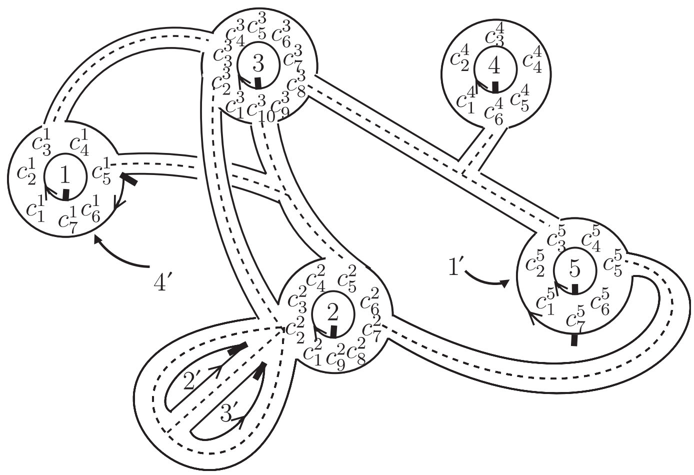
Step I: Construction of α

We need to define maps

$$C_*\mathscr{S}^c(k,l)\otimes \overline{HC^*}(A;A)^{\otimes k}\to \overline{HC^*}(A;A)^{\otimes l}.$$

Let $s\in C_*\mathscr{S}^c(k,l)$, and $f_1,\ldots,f_k\in \overline{HC^*}(A;A)\cong\overline{HC^*}(A;A^*)$. Each $f_i: A^{\otimes n_i}\to A^*$ may be regarded as an element of $(A^*)^{\otimes n_i}\otimes A^*$ and therefore be written as $f_i=(c_1^i,\ldots,c_{n_i}^i;c_{n_i+1}^i)$. In order to define $(\alpha(s))(f_1,\ldots,f_k)\in \overline{HC^*}(A;A)^{\otimes l}$ proceed as follows:

(a) Consider the cyclic Sullivan chord diagram s and place $f_i=(c_1^i,\ldots,c_{n_i}^i;c_{n_i+1}^i)$, for $i=1,\ldots,k$, inside and around the ith input circle of s. To be more precise, at the ith circle of s, start at the input marked point with the last element $c_{n_i+1}^i$ and proceed with c_1^i, c_2^i,..., $c_{n_i}^i$ in the clockwise direction along the input circle; see figure below.

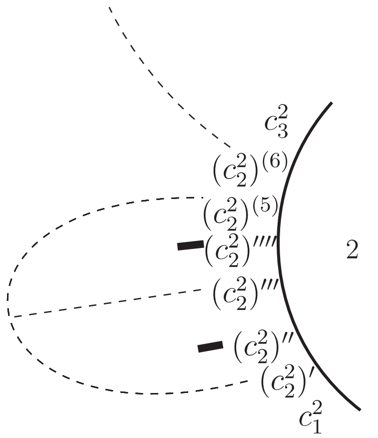


Next, take the sum over all possibilities of placing the output marked points and the chord endpoints on different c_j^i's, while respecting the cyclic ordering of the chord endpoints and output marked points.

(b) If none of the c_j^is has more than one special point attached to it, then go to the next step. Otherwise, if some c_j^is have several special points attached to them, then use the dual of the product $-\cdot -: A\otimes A\to A$ to pull things apart. More precisely, if there are r such things coming together at a c_j^i, we replace c_j^i by

$$(\Delta\otimes id^{\otimes(r-2)})\circ\cdots\circ(\Delta\otimes id)\circ\Delta(c_j^i)=\sum_{(c_j^i)}(c_j^i)'\otimes(c_j^i)''\otimes\cdots\otimes(c_j^i)^{(r)}.$$

Here we have used Sweedler's notation $\Delta: A^*\to A^*\otimes A^*$, $\Delta(c)=\sum_{(c)}c'\otimes c''$. In the proof of Theorem 3.3, c_2^2 is replaced by



If an input marked point is also involved, it can be placed anywhere in between the above linear ordering of chord endpoints and output marked points. The independence from this choice is argued in the next step. Now that some c^i_js are expanded, relabel them so that the last one is again placed at the input marked point. Let us use the same notation $c^i_{n_i+1}$ for the new last element.

(c) Note that now input marked points do not coincide with any output marked points or chord endpoints. We evaluate $c^i_{n_i+1}$ on the unit, to obtain $c^i_{n_i+1}(1) \in k$. In the above picture we obtain $c^1_7(1)$ and $c^2_9(1)$. The ambiguity of where to place the input marked point, which came up in our previous step, is of no issue because the output marked points and chord endpoints are linearly ordered, and the input marked point is evaluated on the unit, 1, which is in the center of A.

(d) We now deal with the c^i_js which are placed at the chord endpoints. The cyclic ordering at each vertex of the chord induces a cyclic ordering of the chord endpoints. In the proof of Theorem 3.3, two of the chords have the endpoints (c^2_5, c^1_5, c^3_{10}), and (c^3_8, c^4_6, c^5_3) up to cyclic permutation.

We will multiply these elements in this cyclic order and evaluate it on the unit. To be more precise, if $c^{i_1}_{j_1}, \ldots, c^{i_r}_{j_r}$ are the endpoints of the chord arranged in the cyclic order, then we obtain the term,

$$\langle \gamma(c^{i_1}_{j_1}) \cdot \ldots \cdot \gamma(c^{i_r}_{j_r}), 1\rangle$$

(see Eq. (1) on p. 8). Here $\gamma : A^* \to A$ is the inverse of the A-bimodule isomorphism $\beta : A \to A^*$.

(e) For each of the l output circles of s, we look at its marked point. Following the orientation of the output circle, we linearly read off the leftover c^i_js (the ones which did not correspond to input marked points or chord endpoints) so that we end with the element at the output marked point. For instance, in the proof of Theorem 3.3, the $1'$st output circle gives rise to the term,

$$(c_1^5, c_2^5, c_9^3, c_6^2, c_6^5; c_7^5).$$

(f) There is an overall sign factor which is obtained in the following way. Note that, using the ordering $f_1, \ldots, f_k$, the c_j^is can be linearly ordered,

$$(c_1^1, \ldots, c_{n_1}^1; c_{n_1+1}^1), \ldots, (c_1^k, \ldots, c_{n_k}^k; c_{n_k+1}^k). \tag{3}$$

The c_j^i's, for $1 \leqslant j \leqslant n_i$, are considered to be of degree 1, whereas $c_{n_i+1}^i$ is regarded as of degree 0. Thus, $f_1 \otimes \cdots \otimes f_k$ has a total degree of $n_1 + \cdots + n_k$. Having this in mind, the operation $\alpha(s)$ can be obtained by the following two steps.
First, the c_j^i's which correspond to chord endpoints are to be evaluated using the inner product. These do not contribute to the output total degree. Similarly, the c_j^i's which correspond to the output marked points change their degree from 1 to 0, because they are to be positioned as the last entry of a Hochschild element. The input marked points $c_{n_i+1}^i$ of degree 0 are evaluated on the unit, and do not change the total degree. We see that $\alpha(s)$ changes the degree by the number of special points on the input circles, minus k. This change of degrees is obtained by applying a tensor product of shift and identity maps to expression (3), where the shift and identity maps have degrees 1 and 0, respectively. In doing so, the usual sign rule applies. That is, whenever something of degree r moves past something of degree s, a sign of $(-1)^{r \cdot s}$ is introduced.
The second step is to rearrange expression (3) according to the combinatorics of the output circles of s. This means that blocks of c_j^i's have to move past other blocks of c_j^i's. We introduce a sign of $(-1)^{r \cdot s}$ for each block of degree r moving past a block of degree s.

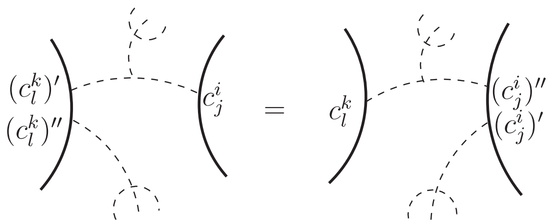
Step II: α is well-defined

To ensure that the above procedure yields a well-defined map, the following checks are in order. We first deal with the fact that output marked points and chord endpoints may slide along chords. For example in the chord diagram in the proof of Theorem 3.3, the $4'$th marked point may be put at c_5^2 instead of c_5^1, while respecting the cyclic ordering of the chord. But we can check that

$$\begin{aligned}
&\sum_{(c_5^1),(c_{10}^3)} \langle \gamma(c_5^2) \cdot \gamma((c_5^1)') \cdot \gamma((c_{10}^3)'), 1\rangle \cdot (c_{10}^3)''(1) \cdot (c_5^1)''(a) \\
&= \langle \gamma(c_5^2) \cdot a \cdot \gamma(c_5^1) \cdot \gamma(c_{10}^3), 1\rangle \\
&= \sum_{(c_5^2),(c_{10}^3)} \langle \gamma((c_5^2)'') \cdot \gamma(c_5^1) \cdot \gamma((c_{10}^3)'), 1\rangle \cdot (c_{10}^3)''(1) \cdot (c_5^2)'(a)
\end{aligned}$$

for all $a \in A$. Also, a chord may slide along another chord, which does not change the outcome because:

$$\begin{aligned}&\sum_{(c_l^k)} \langle \ldots \cdot \gamma(c_j^i) \cdot \gamma((c_l^k)') \cdot \ldots, 1\rangle \cdot \langle \ldots \cdot \gamma((c_l^k)'') \cdot \ldots, 1\rangle \\ &\quad = \sum_{(c_j^i)} \langle \ldots \cdot \gamma((c_j^i)'') \cdot \gamma(c_l^k) \cdot \ldots, 1\rangle \cdot \langle \ldots \cdot \gamma((c_j^i)') \cdot \ldots, 1\rangle\end{aligned}$$



Thus, the example in the proof of Theorem 3.3 yields the following term in expression for $\alpha(s)(f_1, \ldots, f_k)$:

$$\begin{aligned}&\sum_{(c_5^1),(c_2^2),(c_{10}^3),(c_6^4),(c_7^5)} (-1)^{\varepsilon} \left(c_1^5, c_2^5, c_9^3, c_6^2, c_6^5; (c_7^5)'\right) \otimes \left(1_{TA^*}; (c_2^2)''''\right) \otimes \left(1_{TA^*}; (c_2^2)''\right) \\ &\otimes \left(c_6^1, c_1^1, c_2^1, c_4^3, c_5^3, c_6^3, c_7^3, c_1^4, c_2^4, c_3^4, c_4^4, c_5^4, c_4^5, c_8^2, c_1^2, c_4^1, c_1^3, c_3^2, c_4^2; (c_5^1)''\right) \\ &\cdot c_7^1(1) \cdot c_9^2(1) \cdot (c_{10}^3)'(1) \cdot (c_6^4)'(1) \cdot (c_7^5)''(1) \cdot \langle \gamma(c_5^5) \cdot \gamma(c_7^2), 1\rangle \cdot \langle \gamma(c_3^1) \cdot \gamma(c_3^3), 1\rangle \\ &\cdot \langle \gamma(c_8^3) \cdot \gamma((c_6^4)'') \cdot \gamma(c_3^5), 1\rangle \cdot \langle \gamma(c_5^2) \cdot \gamma((c_5^1)') \cdot \gamma((c_{10}^3)''), 1\rangle \\ &\cdot \langle \gamma((c_2^2)') \cdot \gamma((c_2^2)''') \cdot \gamma((c_2^2)^{(5)}), 1\rangle \cdot \langle \gamma((c_2^2)^{(6)}) \cdot \gamma(c_2^3), 1\rangle \qquad (4)\end{aligned}$$

Here, $\varepsilon = (2+4+7+10+12+15+16+21+30+32) + (2 \cdot 20 + 1 \cdot 14 + 1 \cdot 7 + 1 \cdot 19 + 1 \cdot 3 + 4 \cdot 6 + 5 \cdot 6 + 1 \cdot 6 + 1 \cdot 4 + 1 \cdot 2 + 2 \cdot 2) \equiv 0 (mod\ 2)$, and 1_{TA^*} denotes the unit in the tensor algebra TA^*. Note that we could use

$$\begin{aligned}&\sum_{(c_7^5)} (c_7^5)' \cdot ((c_7^5)''(1)) = c_7^5, \\ &\sum_{(c_6^4)} (c_6^4)'(1) \cdot \langle \gamma(c_8^3) \cdot \gamma((c_6^4)'') \cdot \gamma(c_3^5), 1\rangle \\ &\qquad = \langle \gamma(c_8^3) \cdot \gamma(c_6^4) \cdot \gamma(c_3^5), 1\rangle, \\ &\sum_{(c_5^1),(c_{10}^3)} (c_{10}^3)'(1) \cdot \langle \gamma(c_5^2) \cdot \gamma((c_5^1)') \cdot \gamma((c_{10}^3)''), 1\rangle \cdot (c_5^1)'' \\ &\qquad = \langle \gamma(c_5^2) \cdot - \cdot \gamma(c_5^1) \cdot \gamma(c_{10}^3), 1\rangle \in A^*,\end{aligned}$$

where

$$\langle \gamma(c_5^2) \cdot - \cdot \gamma(c_5^1) \cdot \gamma(c_{10}^3), 1\rangle : c \mapsto \langle \gamma(c_5^2) \cdot c \cdot \gamma(c_5^1) \cdot \gamma(c_{10}^3), 1\rangle$$

and other identities to simplify this expression.

It remains to show that this well-defined map, $\alpha : C_*\mathcal{S}^c \to \mathcal{E}nd_{\overline{HC^*}(A;A)}$, respects the differentials, composition, and symmetric group action. It must be clear that the symmetric group action is respected, since it simply has to do with relabelling.

Step III: α respects differentials

The boundary operator, D, on $\mathcal{E}nd_{\overline{HC^*}(A;A)}$ is given by pre and post compositions with the boundary operators on the domain and the range. More precisely, for $\alpha(s) \in \mathcal{E}nd_{\overline{HC^*}(A;A)}(k, l)$, we have the following:

$$
\begin{aligned}
D(\alpha(s))(f_1, \ldots, f_k) = \sum_{j=1}^{k} \alpha(s) \circ (id^{\otimes(j-1)} \otimes \delta \otimes id^{\otimes(k-j)})(f_1, \ldots, f_k) \\
- (-1)^{|s|} \sum_{j=1}^{l} (id^{\otimes(j-1)} \otimes \delta \otimes id^{\otimes(l-j)}) \circ \alpha(s)(f_1, \ldots, f_k),
\end{aligned}
\tag{5}
$$

where $f_i = (c^i_1, \ldots, c^i_{n_i}; c^i_{n_i+1}) \in \overline{HC^*}(A; A)$, and $|s|$ denotes the degree of s. Here, $\delta = \delta_1 + \delta_2$ is the boundary operator on $\overline{HC^*}(A; A)$, which is given by applying the comultiplication $\Delta : A^* \to A^* \otimes A^*$ to all c^i_j's as follows.

$$
\delta_1(c^i_1, \ldots, c^i_{n_i}; c^i_{n_i+1}) = \sum_{j=1}^{n_i} \sum_{(c^i_j)} (-1)^{j-1} \cdot (c^i_1, \ldots, (c^i_j)', (c^i_j)'', \ldots, c^i_{n_i}; c^i_{n_i+1})
$$

$$
\begin{aligned}
\delta_2(c^i_1, \ldots, c^i_{n_i}; c^i_{n_i+1}) = \sum_{(c^i_{n_i+1})} & (-1)^{n_i} (c^i_1, \ldots, c^i_{n_i}, (c^i_{n_i+1})'; (c^i_{n_i+1})'') \\
& + (-1)^{n_i+1} ((c^i_{n_i+1})'', c^i_1, \ldots, c^i_{n_i}; (c^i_{n_i+1})').
\end{aligned}
$$

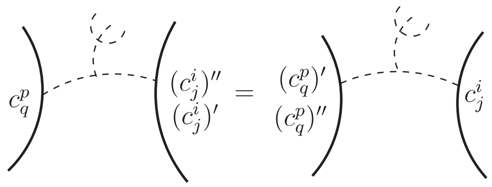
For an $s \in C_*\mathcal{S}^c$, let us compare $D(\alpha(s))$ with $\alpha(\partial(s))$. The right-hand side of Eq. (5) is of the form $S + T_1 + T_2$, where

$$
S = \sum_{j=1}^{k} \alpha(s) \circ (id^{\otimes(j-1)} \otimes \delta \otimes id^{\otimes(k-j)})(f_1, \ldots, f_k),
$$

$$
T_1 = -(-1)^{|s|} \sum_{j=1}^{l} (id^{\otimes(j-1)} \otimes \delta_1 \otimes id^{\otimes(l-j)}) \circ \alpha(s)(f_1, \ldots, f_k),
$$

$$
T_2 = -(-1)^{|s|} \sum_{j=1}^{l} (id^{\otimes(j-1)} \otimes \delta_2 \otimes id^{\otimes(l-j)}) \circ \alpha(s)(f_1, \ldots, f_k).
$$

Each term in S is obtained by first applying Δ to a c^i_j, and then placing special points of α on the outcome in all possible ways as described in *Step* I. In doing so, there are terms in which none, one, or both of the tensor factors of $\Delta(c^i_j) = \sum_{(c^i_j)} (c^i_j)' \otimes (c^i_j)''$ come in contact with the special points. For the purposes of this proof, refer to these terms as S_0, S_1 and S_2, respectively. It is easy to see that $\alpha(\partial(s)) = S_2$. Note that the alternating signs which appears in the definition of the boundary operator corresponds to those arising from the fact that if an element c^i_j is at the rth one of the $|s|+k$ special points on the input circles of s, then $|s|+k-r$ shift maps move over an additional factor of $(c^i_j)'$, giving rise to the appropriate sign. It is also easy to see that $S_0 = -T_1$. It remains to understand what happens with S_1 and T_2. The claim is that $S_1 = -T_2$. Seeing this is a bit less straightforward, since some of the terms in S_1 cancel amongst themselves, whereas other terms cancel with T_2. The following helps to better understand the situation. Consider a simple diagram in which the special points do not coalesce, and let us concentrate on an element in S_1. If an output marked point is placed on one of the factor in Δ, then it corresponds exactly to a term in T_2. If an input marked point is placed on one of the factor in Δ, then it cancels out with a similar term of S_1. This is because for a single input marked point, we are dealing with evaluation on the unit, and the two terms $\sum_{(c^i_j)}((c^i_j)'(1)) \otimes (c^i_j)'' = c^i_j$ and $\sum_{(c^i_j)}(c^i_j)' \otimes ((c^i_j)''(1)) = c^i_j$ cancel. Note that in one of the two expressions the shift map has moved past $(c^i_j)'$, giving rise to a desired negative sign. Now consider the case in which a single chord endpoint is attached to one of the tensor factors, for example, $(c^i_j)''$. Follow the cyclic ordering of the chord's endpoints to go the next chord endpoint which is attached to, let's say, c^p_q. In this case the term involving $(c^i_j)''$ cancels out with a term in the sum where c^p_q is split and the chord is attached to $(c^p_q)'$.



In other words, we use the algebraic fact that

$$\sum_{(c^i_j)}(c^i_j)' \cdot \langle \ldots \cdot \gamma((c^i_j)'') \cdot \gamma(c^p_q) \cdot \ldots, 1\rangle$$
$$= \sum_{(c^p_q)}\langle \ldots \cdot \gamma(c^i_j) \cdot \gamma((c^p_q)') \cdot \ldots, 1\rangle \cdot (c^p_q)''.$$

Note that the signs become opposite when applying Δ to c^i_j on the left side of this equation and moving $(c^i_j)'$ to the spot of c^p_q, instead of applying Δ to c^p_q.

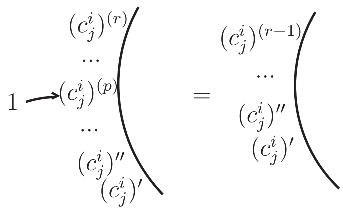
Now, if several special points coalesce, then we can do the same steps as above in the cyclic order specified at this point. One can slide the tensor factors of $\sum_{(c^i_j)}(c^i_j)' \otimes (c^i_j)'' \otimes \cdots \otimes (c^i_j)^{(r)}$ which are not attached to anything from one side to the other in order for them to cancel out.

Step IV: α respects compositions

Let us argue why α respects the composition. Let $s \in C_*\mathcal{S}^c(k,l)$ and $s' \in C_*\mathcal{S}^c(m,k)$, and consider the composition $s \circ s' \in C_*\mathcal{S}^c(m,l)$. Recall that we have to identify the

jth output circle of s' with the jth input circle of s starting at the respective marked points; see p. 4. We need to show that $\alpha(s \circ s') = \alpha(s) \circ \alpha(s')$, where the composition in $\mathscr{E}nd_{\overline{HC^*}(A,A)}$ is given by (2) in Definition 3.2. Thus, for $\alpha(s) \circ \alpha(s')$, we need to apply $\alpha(s)$ to the output $\alpha(s')(f_1, \ldots, f_m) \in \overline{HC^*}(A, A)^{\otimes k}$. Assuming again that each f_i is of the form $f_i = (c^i_1, \ldots, c^i_{n_i}; c^i_{n_i+1}) \in (A^*)^{\otimes n_i} \otimes A^*$, we see that the k tensor factors of $\alpha(s')(f_1, \ldots, f_m)$ consist of c^i_j's following the direction of the outputcircles of s', together with some coefficients, compare (4). We need to apply $\alpha(s)$ to this, which means that output marked points and chords need to be added to the k tensor factors of $\alpha(s')(f_1, \ldots, f_m)$ in all possible ways. Thus one sums over all possibilities of placing output marked points (see item (b) in the proof of Theorem 3.3), and chord endpoints (see item (c) in the proof of Theorem 3.3) on the c^i_j's according to the combinatorics given by s. But this means exactly that we apply chords and output marked points at the points specified by the composition $s \circ s'$. Since everything is graded, and we have to move the same number of elements past each other to obtain the same expression, we also obtain the same overall sign. Notice that this argument also works if several special points coincide at some point, since this only means that the coproduct Δ has to be applied to c^i_j; see item (d) in the proof of Theorem 3.3. The above arguments can be applied to all c^i_j's of the k factors of $\alpha(s')(f_1, \ldots, f_m)$, which are not output marked points. Now, let $\tilde{c} \in A^*$ represent one of the output marked points. The definition of $\alpha(s)$ in item (a) on p. 11 requires to apply the unit 1 to this element $\tilde{c}$. Note that $\tilde{c}$ might either be a factor of some coproduct—such as $(c^5_7)'$, $(c^2_2)''''$, $(c^2_2)''$ and $(c^1_5)''$ in Eq. (4)—or not.In the first case, we can completely eliminate this marked point by using the algebraic fact

$$
\begin{aligned}
&\sum_{(c^i_j)} (c^i_j)' \otimes (c^i_j)'' \otimes \cdots \otimes (c^i_j)^{(p)}(1) \otimes \cdots \otimes (c^i_j)^{(r)} \\
&\quad = \sum_{(c^i_j)} (c^i_j)' \otimes (c^i_j)'' \otimes \cdots \otimes (c^i_j)^{(r-1)}
\end{aligned}
$$



In the second case, we apply the unit 1 to some $\tilde{c}=c^i_j$, where $1 \leqslant j \leqslant n_i$. But in the normalized Hochschild complex, we have $f_i(\ldots, 1, \ldots) = 0$, or $c^i_j(1) = 0$. Thus, the composition vanishes. This is consistent with the fact that $s \circ s' = 0$, since the dimension of $s \circ s'$ is less than the sum of the dimensions of s and s'; see p. 5.

We have shown that the action of $\alpha(s) \circ \alpha(s')$ is the same as the action of $\alpha(s \circ s')$, and this completes the proof of the theorem. $\square$

4. The commutative case

Throughout this section A denotes an associative, commutative, and unital algebra, which is endowed with a non-degenerate invariant inner product. We describe how the PROP of cyclic Sullivan chord diagrams, $C_*\mathscr{S}^c$, can be enlarged to include orientation-reversing chords in the action. Thickening such a chord gives rise to a non-orientable surface with boundary. This enlarged PROP, denoted by $C_*\mathscr{S}$, will then act on the Hochschild complex of the algebra A.

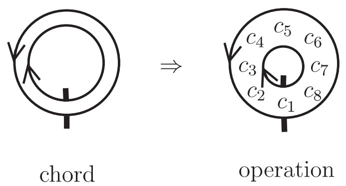
Observations 4.1. (i) Define the *orientation-reversing* operation

$$\sim: HC^*(A; A) \to HC^*(A; A)$$
$$\sim: f \mapsto \tilde{f}$$
$$\tilde{f}(a_1, a_2, \ldots, a_{n-1}, a_n) := (-1)^{\frac{n(n+1)}{2}} \cdot f(a_n, a_{n-1}, \ldots, a_2, a_1)$$

One can check that $(\delta(\tilde{f}) - \widetilde{\delta(f)})(a_1, \ldots, a_{n+1})$ is equal to

$$\sum_{j=1}^{n} \pm f(a_{n+1}, \ldots, a_j a_{j+1} - a_{j+1} a_j, \ldots, a_1)$$
$$\pm (a_1 \cdot f(a_2, \ldots, a_{n+1}) - f(a_2, \ldots, a_{n+1}) \cdot a_1)$$
$$\pm (a_{n+1} \cdot f(a_1, \ldots, a_n) - f(a_1, \ldots, a_n) \cdot a_{n+1}).$$

Since A is commutative, this expression vanishes. This means that for commutative A the map $\sim$ is a chain map of the Hochschild complex into itself. The operation $\sim$ can be obtained from the following chord diagram, where we insert a string of elements in one direction, and read them off in the opposite direction. We refer to this diagram as the orientation-reversing chord diagram.



Note that this chord diagram is a closed element in the complex of chord diagrams, as defined in Section 2.

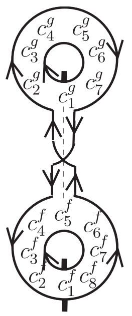
(ii) Let's look at the brace operation $*$ from p. 6. For $f, g \in HC^*(A; A)$ we have

$$(f * g)(a_1, \ldots, a_n) = \sum_k \pm f(a_1, \ldots, a_k, g(a_{k+1}, \ldots, a_{k+l}), a_{k+l+1}, \ldots, a_n).$$

We now want to allow reversing of orientations, as described in (i). For example the operation $\tilde{f} * g$, defined below, is also legitimate. In this case $(\tilde{f} * g)(a_1, \ldots, a_n)$ is equal to

$$\sum_k \pm f(a_n, \ldots, a_{k+l+1}, g(a_{k+1}, \ldots, a_{k+l}), a_k, \ldots, a_1).$$

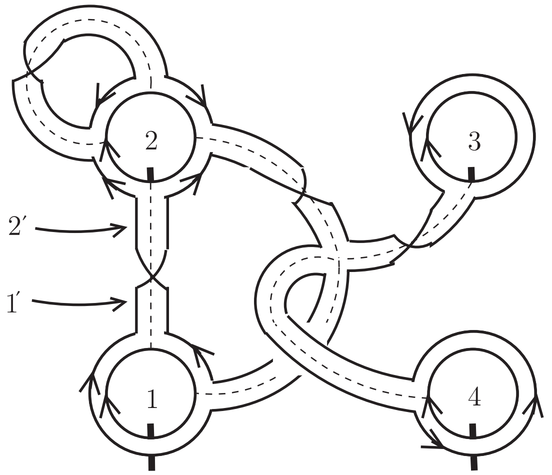
Note that the elements plugged into f are reversed, while those plugged into g have preserved their linear ordering. The following figure shows that this phenomenon can be expressed by considering diagrams with twisted chords.



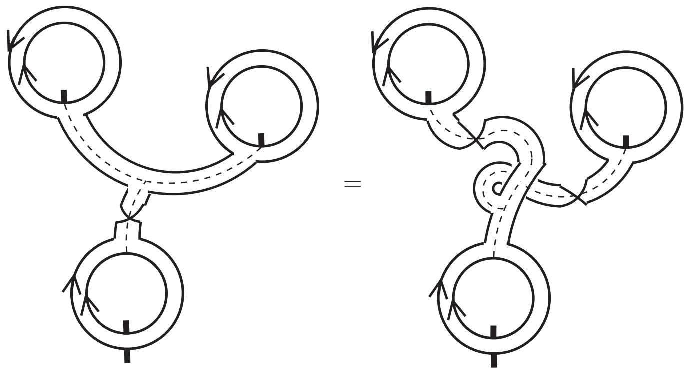
These two observations demonstrate all the new features of diagrams describing orientation-reversing operations. Chord diagrams with possible twisted chords form a PROP which acts on the normalized Hochschild cochain complex of A. The relevant definition and its application to the normalized Hochschild cochain complex will occupy the rest of the paper.

Definition 4.2 (*Sullivan chord diagram*). A *Sullivan chord diagram* is a generalization of a cyclic Sullivan chord diagram, where chords may have twists in them and the orientation of output circles may be arbitrary.

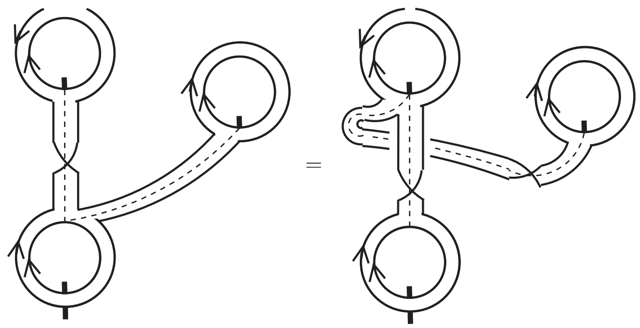
Note that, when moving along an output circle, one may alternate between going in the direction compatible with those of the input circles, and the opposite direction. This is shown in the below figure, where the direction of $1'$st output circle is compatible with that of the 1st input circle, but in opposition to that of the 2nd input circle. A similar remark applies to going along the input circles, as seen for example the 1st input circle.



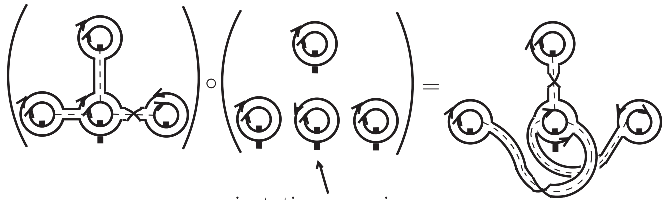
We also remark that Sullivan chord diagrams are considered up to abstract isomorphism of the thickened surfaces respecting all labelling and orientations (see figure below).



In particular, a chord with two adjacent twists is identified with a chord without a twist. Also, note that the relations among diagrams, such as sliding along a chord, now have to respect the twists of that chord (see figure below).



These more general diagrams are made into a PROP, denoted by $C_*\mathscr{S}$, similarly to the case of $C_*\mathscr{S}^c$ described in Section 2. In fact, the tensor product, the symmetric group action and the differential are exactly the same. As for the composition, the following comments are in order.



Consider the situation of a composition $s \circ s'$, where the orientation of an output circle of s' is opposite to that of the corresponding input circle of s. When applying the definition of the composition from Section 2, we need to follow the orientation of the output circle and identify it with that of the corresponding input circle. In the above picture, this is achieved by flipping an input circle, which has introduced twists in some of the chords. In general, while sewing input to output, one needs to give chords an extra twist, if the orientations do not match.

Theorem 4.3. *Let A be a finite dimensional, unital, associative, and commutative algebra endowed with a non-degenerate and invariant inner product. Then, the normalized Hochschild cochain complex of A is an algebra over the PROP, $C_*\mathcal{S}$, of Sullivan chord diagrams.*

Proof. The description of the map $\overline{\alpha} : C_*\mathcal{S} \to \mathcal{E}nd_{\overline{HC^*}(A;A)}$ is identical to that of the previously established action $\alpha : C_*\mathcal{S}^c \to \mathcal{E}nd_{\overline{HC^*}(A;A)}$ in the associative case. It remains to show that $\overline{\alpha}$ is a map of operads. One can see that $\overline{\alpha}$ resects composition for the same reasons α did. In fact, commutativity of A does not play a role in this. Commutativity of A, however, plays an important role in showing that $\overline{\alpha}$ respect the differentials. Recall that for $s \in C_*\mathcal{S}$, formula (5) describes the differential $D(\alpha(s))$. There are two cases to consider. In case there is at most one binding tensor factor of $\Delta(c^i_j) = \sum_{(c^i_j)} (c^i_j)' \otimes (c^i_j)''$, the terms in (5) cancel each other. This is due to the new feature of commutativity of A, as seen in observation [4.1(i)]. In case both tensor factors of $\Delta(c^i_j)$ are bound, we obtain the terms which correspond to $\alpha(\partial(s))$. □

Note that the chord associated to the orientation-reversal $\sim$ squares to the identity, $(\sim)^2 = id$. Therefore, $\overline{HC^*}(A;A)$ decomposes into the eigenspaces $\overline{HC^*}(A;A)_+ \oplus \overline{HC^*}(A;A)_-$, where

$$\overline{HC^*}(A;A)_+ = span\{c_1 \otimes \cdots \otimes c_n \otimes a + (-1)^\varepsilon c_n \otimes \cdots \otimes c_1 \otimes a\},$$
$$\overline{HC^*}(A;A)_- = span\{c_1 \otimes \cdots \otimes c_n \otimes a - (-1)^\varepsilon c_n \otimes \cdots \otimes c_1 \otimes a\},$$

where $c_i \in A^*$, $a \in A$, and $\varepsilon = \frac{n(n+1)}{2}$. The operator Δ maps each eigenspace into the other, i.e. $\Delta\left(\overline{HC^*}(A;A)_\pm\right) \subset \overline{HC^*}(A;A)_\mp$. In the notation of Example 2.3, $\sim$ anticommutes with Δ, i.e. $\sim \circ \Delta = -\Delta \circ \sim$. Moreover, we have $\sim \circ \smile = \smile \circ \tau_2 \circ (\sim \otimes \sim)$ and $(\sim \otimes \sim) \circ \tau_2 \circ \vee_0 = \vee_0 \circ \sim$. Since $\sim$ commutes with the Hochschild boundary operator, we have:

Corollary 4.4. *Under the assumptions of Theorem 4.3, the Hochschild cohomology of A is a Frobenius algebra, which is endowed with a compatible BV operator, Δ, and an involution $\sim$. The operator Δ maps each eigenspace of $\sim$ into the other, i.e. $\Delta(HH^*(A;A)_\pm) \subset HH^*(A;A)_\mp$, where $HH^*(A;A)_\pm$ are the ± 1 eigenspaces of $\sim$. The map $\sim$ is both an anti-algebra and an anti-coalgebra map. That is to say $\widetilde{f \smile g} = \widetilde{g} \smile \widetilde{f}$, and $\vee_0(\widetilde{f}) = \sum_{(f)} \widetilde{f''} \otimes \widetilde{f'}$, where $\vee_0(f) = \sum_{(f)} f' \otimes f''$.*

Acknowledgements

We would like to thank James Stasheff and Dennis Sullivan for their constructive comments.

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