

Higher Hochschild Homology, Topological Chiral Homology and Factorization Algebras

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Abstract: We study the higher Hochschild functor, factorization algebras and their relationship with topological chiral homology. To this end, we emphasize that the higher Hochschild complex is a functor $sSet_\infty \times CDGA_\infty$ where $sSet_\infty$ and $CDGA_\infty$ are the $(\infty, 1)$ -categories of simplicial sets and commutative differential graded algebras, and give an axiomatic characterization of this functor. From the axioms, we deduce several properties and computational tools for this functor. We study the relationship between the higher Hochschild functor and factorization algebras by showing that, in good cases, the Hochschild functor determines a constant commutative factorization algebra. Conversely, every constant commutative factorization algebra is naturally equivalent to a Hochschild chain factorization algebra. Similarly, we study the relationship between the above concepts and topological chiral homology. In particular, we show that on their common domains of definition, the higher Hochschild functor is naturally equivalent to topological chiral homology. Finally, we prove that topological chiral homology determines a locally constant factorization algebra and, further, that this functor induces an equivalence between locally constant factorization algebras on a manifold and (local system of) E_n -algebras.

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1. Introduction

In this paper, we study the higher Hochschild chain¹ complex $CH_{X_\bullet}^\bullet(A)$, functorially assigned to a simplicial set X_\bullet (or a topological space), and a commutative differential graded algebra (CDGA) A , from an axiomatic point of view. Recently, motivated by topological quantum field theories, several concepts integrating (higher) categories of spaces or manifolds with those of algebras of different types have arisen. We also study the relationship between higher Hochschild chains, factorization algebras [8, 9] and topological chiral homology [24, 25]. Higher Hochschild homology was first introduced by Pirashvili in [28]. The higher Hochschild complexes (as well as other aforementioned concepts) are a generalization of the classical Hochschild complex. In fact, for the case of the standard simplicial set model $X_\bullet = S_\bullet^1$ for the circle, $CH_{S_\bullet^1}^\bullet(A)$ reduces to the standard Hochschild complex $CH_\bullet(A) = A^{\otimes \bullet+1}$, see [17, 18].

In contrast with most other generalizations, higher Hochschild chains are defined over *any* (simplicial set model of a) space and not only (stratified) manifolds. However, this forces us to restrict our attention to CDGAs or at best to E_∞ -algebras. More precisely, the higher Hochschild chains form a bifunctor $CH : sSet \times CDGA \rightarrow CDGA$ from the categories of simplicial sets and differential graded commutative algebras to the latter category. The functoriality with respect to spaces (and not merely manifold embeddings) is a key feature which allows us to derive algebraic operations on the higher Hochschild chain complexes from maps of topological spaces. For instance, it was crucially used to study the Hodge decomposition of Hochschild homology (Pirashvili [28], also see [26] in the topological context) or to give and study models of (higher) string topology [13, 14]. Also, its underlying combinatorial properties allow a generalization of Chen's iterated integral [14]. Higher Hochschild is also a convenient setting to study holonomy of (higher) gerbes (for instance see [33]) or compute the observables of classical topological field theories, see [8] and Sect. 4.2. The higher Hochschild homology satisfies many axioms similar to those of Eilenberg–Steenrod for singular homology: naturality in each variable, commutations with coproducts in both variables, homotopy invariance and the dimension axiom, see Corollary 2.

However, to fully appreciate the higher Hochschild functor, one needs to go beyond mere homology and consider the higher Hochschild chains in a *derived setting*, which allows one to formulate the analogue of the *excision axiom*. This axiom, reminiscent

¹ Note, that in this paper we are using a *cohomological* grading for our differential graded modules, see Convention (7) on page 7. Also, this paper only deals with Hochschild chains and Hochschild homology, and never with Hochschild cochains or Hochschild cohomology.

of the locality axioms of topological field theories, asserts that Hochschild chains map the homotopy pushout of simplicial sets to the derived tensor product of algebras, *i.e.* homotopy pushout of CDGAs. This gluing property together with the homotopy invariance allow one to build many examples of Hochschild chain complexes and to do computations as demonstrated in [14]. Further, such an enhancement is needed in order to correctly compare the higher Hochschild functor with more sophisticated concepts, such as topological chiral homology, which naturally lies in a homotopical setting. More precisely, we interpret the higher Hochschild chains as a (derived) bifunctor from the $(\infty, 1)$ -categories $sSet_\infty$ of simplicial sets and $CDGA_\infty$ of CDGAs, which are suitable localizations of the categories of simplicial sets and CDGAs, with respect to (weak) homotopy equivalences and quasi-isomorphisms. This framework (instead of simply homology) is also needed to keep track of the topology of topological spaces modeled by the simplicial sets; for instance the usual Hochschild complex $CH_*(A)$ interpreted in an $(\infty, 1)$ -category retains a circle action governing cyclic homology as shown in [25, 32]. Here, following Rezk and Lurie [25, 29], an $(\infty, 1)$ -category means a complete Segal space. In our context, the $(\infty, 1)$ -categories we considered are obtained by a Dwyer–Kan localization process from standard model categories, though the results of this paper should not depend on the particular chosen approach to $(\infty, 1)$ -categories, see also Remark 1.

Our first main result is the following theorem.

Theorem 1. *The Hochschild chains lift as a functor of $(\infty, 1)$ -categories $CH : sSet_\infty \times CDGA_\infty \rightarrow CDGA_\infty$ which satisfies the following axioms*

1. **value on a point:** *there is a natural equivalence of CDGAs $CH_{pt}^\bullet(A) \cong A$.*
2. **monoidal:** *there are natural equivalences of CDGAs*

$$CH_{\coprod X_i}^\bullet(A) \cong \bigotimes CH_{X_i}^\bullet(A)$$

3. **homotopy gluing/pushout:** *CH sends homotopy pushout in $sSet_\infty$ to homotopy pushout in $CDGA_\infty$, *i.e.* there is a natural equivalence of CDGAs*

$$CH_{X_\bullet \cup_{Z_\bullet}^h Y_\bullet}^\bullet(A) \cong CH_{X_\bullet}^\bullet(A) \otimes_{CH_{Z_\bullet}^\bullet(A)} CH_{Y_\bullet}^\bullet(A).$$

Furthermore, the above axioms actually *define* the (derived) higher Hochschild chains: indeed our second main result, Theorem 2 can be rephrased as

Theorem 2. *The Hochschild chains is the unique bifunctor $sSet_\infty \times CDGA_\infty \rightarrow CDGA_\infty$ satisfying the axioms (1), (2), (3) in Theorem 1.*

These two results actually follow from the fact that $CDGA$ is tensored over simplicial sets and the general formalism of $(\infty, 1)$ -categories as in [22, 24] and allow one to interpret the Hochschild functor as a (derived) mapping stack in the context of [30], see Corollary 15. We also show that the derived Hochschild functor $CH : sSet_\infty \times CDGA_\infty \rightarrow CDGA_\infty$ has many good formal properties: for instance it commutes with finite (homotopy) colimits in both arguments and with finite products of simplicial spaces (Corollary 5 and Proposition 5). Further, the locality axioms lead to an Eilenberg–Moore spectral sequence computing the higher Hochschild homology (Corollary 3).

We also deal with the pointed versions of higher Hochschild chains, which allows one to define Hochschild chains over a pointed simplicial set X_\bullet of a CDGA A with coefficient in an A -module M and establish similar results for this theory.

By homotopy invariance, we can define $CH_X^\bullet(A)$ for a topological space X , generalizing the concept for a simplicial set X_\bullet , in such a way that all of the above properties still hold. With this, we can now offer interpretations of $CH_X^\bullet(A)$ in various contexts. First, in Sect. 4, we use these properties to give an interpretation of Hochschild chains over spaces of a CDGA A as a factorization algebra in any dimension. The concept of *factorization algebras* (see [8, 9]) is inspired by Topological Quantum Field Theory, in which they appear naturally to encode observables. They were inspired by the work of Beilinson and Drinfeld [3] (in an algebraic-geometry framework). Roughly speaking a factorization algebra \mathcal{F} is a rule which (covariantly) associates cochain complexes to open subsets of a space X together with multiplications

$$\mathcal{F}(U_1) \otimes \cdots \otimes \mathcal{F}(U_n) \rightarrow \mathcal{F}(V)$$

for any family of pairwise disjoint open subsets of an open set V in X . It should satisfy a “cosheaf-like” condition, meaning that $\mathcal{F}(V)$ can be computed by Čech complexes indexed on nice enough covers, called factorizing covers, see [8] and Sect. 4.2. The (derived) global sections of a factorization algebra \mathcal{F} is also called the factorization homology of \mathcal{F} and is denoted $HF(\mathcal{F}, X)$.

In this context we prove that the higher Hochschild chain functor defines a commutative factorization algebra $\mathcal{CH}_X(A)$, if X admits a good cover whose factorization homology is precisely the derived Hochschild chains $CH_X^\bullet(A)$.

Theorem 4. *Let X be a topological space with a factorizing good cover and A be a CDGA. Assume further that there is a basis of open sets in X which is also a factorizing good cover. Then the assignment $\mathcal{CH}_X : U \mapsto CH_U^\bullet(A)$ is a factorization algebra on X .*

In particular, this applies when X is a manifold. Further, we prove that any factorization algebra for which $\mathcal{F}(U)$ (for contractible U) is naturally equivalent to a CDGA A is canonically equivalent to $\mathcal{CH}_X(A)$.

Corollary 10. *Let X be a topological space with a sufficiently nice cover, let A be a CDGA, and let \mathcal{F} be a strongly constant factorization algebra on X of type A . Then there is a natural equivalence of factorization algebras $\mathcal{F} \cong \mathcal{CH}_X(A)$. In particular, there is a natural equivalence $HF(\mathcal{F}) \cong CH_X^\bullet(A)$ in $k\text{-Mod}_\infty$.*

In Sect. 5, we establish a relationship between the *topological chiral homology* functor defined by Lurie [24, 25] and both the higher Hochschild functor and factorization algebras. To obtain a comparison between these functors, it is important to note that they are defined in two different settings with a common intersection. Topological chiral homology, denoted $\int_M A$, is defined for any E_n -algebra A (where E_n is an operad equivalent to the little cubes in dimension n) and an m -dimensional manifold M , $m \leq n$, such that $M \times D^{n-m}$ is framed (we say M is n -framed). Further $\int_M A$ is an E_{n-m} -algebra which is also a module over the E_{n-m+1} -algebra $\int_{\partial M} A$. Topological chiral homology can be interpreted as an invariant of framed manifolds produced by an extended (∞, n) -Topological Field Theory in the sense of [25]; the theory in question takes values in an (∞, n) -category of E_n -algebras whose n -morphisms are (homotopy types) of chain complexes. Note that topological chiral homology depends on and comes with a choice of a sequence of maps of operads,

$$E_1 \hookrightarrow E_2 \hookrightarrow \cdots \hookrightarrow E_n \hookrightarrow \cdots \twoheadrightarrow Com$$

which allows one to interpret a CDGA as an E_n -algebra for any n . When A is a CDGA, things simplify greatly, and we can give a simple description of $\int_M A$ in terms of the higher Hochschild complex of A . Using excision for topological chiral homology, see Proposition 11, we prove, in a rather geometric way:

Theorem 5. *Let M be a manifold endowed with a framing of $M \times D^k$ and A be a differential graded commutative algebra viewed as an E_{m+k} -algebra. Then topological chiral homology of M with coefficients in A , denoted by $\int_M A$ is equivalent to $CH_M^\bullet(A)$ viewed as an E_k -algebra.*

In other words, topological chiral homology and higher Hochschild chains coincide on their common intersection for an n -framed manifold M , and a CDGA A . As an immediate corollary, in that case, $\int_M A$ is independent of the n -framing, see Sect. 5.4.

The relation between factorization algebras, derived higher Hochschild chains, and topological chiral homology for CDGAs can be pushed further. Indeed, the data of an E_n -algebra are equivalent to those of a locally constant factorization algebra in \mathbb{R}^n [9, 24], see Proposition 10. Further, the assumptions of having an E_n -algebra and a framed manifold to define topological chiral homology can be replaced by the one of having a suitable (kind of) cosheaf of E_n -algebras on an n -dimensional manifold N . Such a cosheaf is called an \mathbb{E}_N^\otimes -algebra [24] and is also inspired by the work of Beilinson-Drinfeld [3]. The techniques developed to compare Hochschild chains with factorization algebras and topological chiral homology leads to Theorem 6 which can be rephrased as

Theorem 6. *Let M be a manifold of dimension n .*

1. *Topological chiral homology defines a natural $(\infty, 1)$ -functor \mathcal{TC}_M from the category of $\mathbb{E}_{M \times \mathbb{R}^d}^\otimes$ -algebras to the category of locally constant factorization algebras on M with value in E_d -algebras, such that $\int_M A \cong HF(\mathcal{TC}_M, M)$.*
2. *The functor $\mathcal{TC}_M(A)$ is an equivalence.*

Let us outline the philosophy intertwining the different concepts studied here. Given an n -framed manifold M of dimension m (i.e. $M \times \mathbb{R}^{n-m}$ is framed), and an E_n -algebra A , we can form the topological chiral homology $\int_M A$ (or equivalently consider factorization algebra homology), which can be thought of as a colimit of tensor products of A indexed by balls in the manifold. Now, if we embed $M \times \mathbb{R}^{n-m}$ in $M \times \mathbb{R}^{n-m+1}$ equipped with the induced framing, one can form $\int_M B$ for an E_{n+1} -algebra. But two different framings of $M \times \mathbb{R}^{n-m}$ may become equivalent after the embedding. Since a CDGA C is an E_k -algebra (as well as an \mathbb{E}_M^\otimes -algebra) for all k , $\int_M C$ should not be able to distinguish different framings. Since manifolds embed in euclidean spaces, we further see that $\int_M C$ should make sense for *any* manifold. Note that constant factorization algebra can be pulled back along open immersions and pushed forward any map. This hints that any deformation retract of a manifold should also have a well defined topological chiral homology (with value in a C) equivalent to the one of the manifold. All of this suggests that, for CDGAs, topological chiral homology may be extended to any CW-complex and is a homotopy invariant, which is precisely realized by the derived higher Hochschild functor. Said otherwise, higher Hochschild is the “limit” for n going to ∞ of topological chiral homology defined as an invariant of manifolds of dimension n .

One of the emerging patterns here is that there is a balance to keep between the manifolds and the algebraic structure needed to produce a (derived) invariant. For instance, in order to consider E_n -algebras, one needs to consider only at most n -dimensional manifolds (possibly with extra structure such as a framing). In particular, working with only

associative algebras restricts attention to manifolds of dimension 1. At the opposite side of the spectrum, restricting to CDGAs allows one to build and study *explicit* examples in a much easier way and to compute them when adding the usual Rational Homotopy techniques to the axiomatic properties satisfied by the theory.

We choose to work with commutative differential graded algebras since we are mainly interested in the characteristic zero case. However, it is also possible to work with simplicial commutative algebras, and all of our results should make sense in this setting. Simplicial commutative algebras are better behaved if one wants to deal with positive characteristic.

2. Preliminary Definitions and Notation

In this section we recall some standard definitions and constructions.

Conventions:

1. We fix a ground field k of characteristic zero. The $(\infty, 1)$ -category of differential graded k -modules (*i.e.* complexes) will be denoted $k\text{-Mod}_\infty$.
2. The (naïve) categories of simplicial sets and of commutative differential graded algebras will be respectively denoted by $s\text{Set}$ and $CDGA$. The category of commutative graded algebra will be denoted CGA . Unless otherwise stated, all algebras will be assumed to be *unital*.
3. We will simply refer to commutative differential graded algebras as CDGAs.
4. The $(\infty, 1)$ -categories of simplicial sets and commutative differential graded algebras will be respectively denoted by $s\text{Set}_\infty$ and $CDGA_\infty$.
5. Let $n \geq 1$ be an integer. By an E_n -algebra we mean an algebra over an E_n -operad. Unless otherwise stated, we work in the context of operads of differential graded k -modules or ∞ -operads in $k\text{-Mod}_\infty$. We will write $E_n\text{-Alg}_\infty$ for the $(\infty, 1)$ -category of E_n -algebras.
6. We work with a cohomological grading (unless otherwise stated) for all our (co)homology groups and graded spaces, even when we use subscripts to denote the grading. In particular, all differentials are of degree +1, of the form $d : A^i \rightarrow A^{i+1}$ and the homology groups $H_i(X)$ of a space X are concentrated in non-positive degree.
7. We will denote by $CH_{X_\bullet}^n(A)$ the *Hochschild chain complex* over X_\bullet with value in A of total degree n . This Hochschild chain complex was noted differently in the papers [13, 14]. We choose this notation in order to put emphasis on the *covariance* of the Hochschild chain functor with respect to X_\bullet and the fact that we are considering cohomological degree.

2.1. Simplicial sets. Denote by Δ the category whose objects are the ordered sets $[k] = \{0, 1, \dots, k\}$, and morphisms $f : [k] \rightarrow [l]$ are non-decreasing maps $f(i) \geq f(j)$ for $i > j$. In particular, we have the morphisms $\delta_i : [k-1] \rightarrow [k]$, $i = 0, \dots, k$, which are injections that miss i and we have surjections $\sigma_j : [k+1] \rightarrow [k]$, $i = 0, \dots, k$, which send j and $j+1$ to j .

A *simplicial set* is by definition a contravariant functor from Δ to the category of sets *Sets* or written as a formula, $Y_\bullet : \Delta^{op} \rightarrow \text{Sets}$. Denote by $Y_k = Y_\bullet([k])$, and call its elements simplices. The image of δ_i under Y_\bullet is denoted by $d_i := Y_\bullet(\delta_i) : Y_k \rightarrow Y_{k-1}$, for $i = 0, \dots, k$, and is called the i th face. Similarly, $s_i := Y_\bullet(\sigma_i) : Y_k \rightarrow Y_{k+1}$, for $i = 0, \dots, k$, is called the i th degeneracy. An element in Y_k is called a degenerate simplex, if it is in the image of some s_i , otherwise it is called non-degenerate.

A simplicial set is said to be *finite* if Y_k is finite for every object $[k] \in \Delta$. A *pointed* simplicial set is a contravariant functor into the category \mathbf{Sets}_* of pointed finite sets, $Y_\bullet : \Delta^{op} \rightarrow \mathbf{Sets}_*$. In particular, each $Y_k = Y_\bullet([k])$ has a preferred element called the basepoint, and all differentials d_i and degeneracies s_i preserve this basepoint.

A morphism of (finite or not, pointed or not) simplicial sets is a natural transformation of functors $f_\bullet : X_\bullet \rightarrow Y_\bullet$. Thus f_\bullet is given by a sequence of maps $f_k : X_k \rightarrow Y_k$ (preserving the basepoint in the pointed case), which commute with the faces $f_k d_i = d_i f_{k+1}$, and degeneracies $f_{k+1} s_i = s_i f_k$ for all $k \geq 0$ and $i = 0, \dots, k$.

One of the most important construction for us is the pushout.

Definition 1. Let X_\bullet , Y_\bullet , and Z_\bullet be simplicial sets, and let $f_\bullet : Z_\bullet \rightarrow X_\bullet$ and $g_\bullet : Z_\bullet \rightarrow Y_\bullet$ be maps of simplicial sets. We define the wedge $W_\bullet = X_\bullet \cup_{Z_\bullet} Y_\bullet$ of X_\bullet and Y_\bullet along Z_\bullet as the simplicial space given by $W_k = (X_k \cup Y_k) / \sim$, where \sim identifies $f_k(z) = g_k(z)$ for all $z \in Z_k$. The face maps are defined as $d_i^{W_\bullet}(x) = d_i^{X_\bullet}(x)$, $d_i^{W_\bullet}(y) = d_i^{Y_\bullet}(y)$ and the degeneracies are $s_i^{W_\bullet}(x) = s_i^{X_\bullet}(x)$, $s_i^{W_\bullet}(y) = s_i^{Y_\bullet}(y)$ for any $x \in X_k \hookrightarrow W_k$ and $y \in Y_k \hookrightarrow W_k$. It is clear that W_\bullet is well-defined and there are simplicial maps $X_\bullet \xrightarrow{i_\bullet} W_\bullet$ and $Y_\bullet \xrightarrow{j_\bullet} W_\bullet$.

If X_\bullet is a pointed simplicial set, then we can make W_\bullet into a pointed simplicial set by declaring the basepoint to be the one induced from the inclusion $X_\bullet \rightarrow W_\bullet$. (Note that this is in particular the case, when X_\bullet , Y_\bullet , Z_\bullet , f_\bullet and g_\bullet are in the pointed setting.)

2.2. Commutative differential graded algebras. We let $CDGA$ be the category of commutative differential graded algebras (over the characteristic zero field k). We do not assume the underlying chain complexes of our algebras to be bounded, since in practice, it happens that one has to consider the Hochschild chains of de Rham forms on a space, which is generally \mathbb{Z} -graded. We follow the approach of [30, Chapter 1.1] and [16] for the model category properties of $CDGA$ and modules over CDGAs. Recall from [19, Section 2.3], that there is a standard cofibrantly generated closed model category structure on the category of unbounded chain complexes for which fibrations are epimorphisms and (weak) equivalences are quasi-isomorphisms. It is further a symmetric monoidal model category with respect to the tensor products of chain complexes.

Since we work in characteristic zero, there is a standard closed model category structure on $CDGA$ [16, Theorem 4.1.1] as well, for which fibrations are epimorphisms and (weak) equivalences are quasi-isomorphisms (of CDGAs). The category $CDGA$ also has a monoidal structure given by the tensor product (over the ground field k) of differential graded commutative algebras, which makes $CDGA$ a symmetric monoidal model category. Note that since k is assumed to be a field, this monoidal structure is given by an exact bifunctor.

Also note that $CDGA$ is simplicially enriched. Indeed, given $A, B \in CDGA$, we can form $\text{Map}_{CDGA}(A, B)$ the simplicial set $[n] \mapsto \text{Hom}_{CDGA}(A, B \otimes \Omega^*(\Delta^n))$ (where $\Omega^*(\Delta^n)$ is the CDGA of forms on the n -dimensional standard simplex).

For any CDGA A , one can consider its category of *differential graded (left) modules*, that we will denote by $A\text{-Mod}$. Again it has a natural model category structure with fibrations being epimorphisms and weak equivalences being quasi-isomorphisms. Further all assumptions in [30, Chapter 1.1] are satisfied. In particular, the tensor product of A -modules makes $A\text{-Mod}$ a symmetric monoidal model category (in the sense of [19]) such that the functor $M \otimes_A$ —preserves weak equivalences when M is cofibrant. Moreover, for any CDGA A , the category $A\text{-}CDGA$ of differential graded commutative

A -algebra, in other words commutative monoid objects in $A\text{-Mod}$, has a natural structure of proper model category such that, for any cofibrant A -algebra B , the base change functor $B \otimes_A - : A\text{-Mod} \rightarrow B\text{-Mod}$ preserves weak equivalences [30, Chapter 1.1].

2.3. Dwyer–Kan localization and $(\infty, 1)$ -categories. The $(\infty, 1)$ -categories that we are concerned about in this paper arise from model categories structures via the Dwyer–Kan localization turning them into simplicial categories. Indeed simplicial categories are models for $(\infty, 1)$ -categories [5]. We now explain briefly how one gets $(\infty, 1)$ -categories out of model categories such as those considered in Sects. 2.1 and 2.2 above.

Following [25, 29], by an $(\infty, 1)$ -category we mean a *complete Segal space*. Rezk has shown that the category of simplicial spaces has a (simplicial closed) model structure, denoted \mathcal{CSeSp} such that a complete Segal space is precisely a fibrant object for this model structure [29, Theorem 7.2]. Note that there is also a (simplicial closed) model category structure, denoted \mathcal{SeSp} , on the category of simplicial spaces such that a fibrant object in the \mathcal{SeSp} structure is precisely a Segal space. We let $\mathbb{R} : \mathcal{SeSp} \rightarrow \mathcal{CSeSp}$ be a fibrant replacement functor. Rezk [29] has defined a completion functor $X_\bullet \rightarrow \widehat{X}_\bullet$ which, to a Segal space, associates an equivalent complete Segal space. Thus, the composition $X_\bullet \mapsto \widehat{\mathbb{R}(X_\bullet)}$ gives a (fibrant replacement in the model category \mathcal{CSeSp}) functor $L_{\mathcal{CSeSp}}$ from simplicial spaces to complete Segal spaces.

It remains to explain how to go from a model category to a simplicial space. The standard key idea is to use Dwyer–Kan localization. Let \mathcal{M} be a model category and \mathcal{W} be its subcategory of weak-equivalences. We denote $L^H(\mathcal{M}, \mathcal{W})$ its *hammock localization*, see [10]. One of the main property of $L^H(\mathcal{M}, \mathcal{W})$ is that it is a simplicial category and that the (usual) category $\pi_0(L^H(\mathcal{M}, \mathcal{W}))$ is the homotopy category of \mathcal{M} . Further, every weak equivalence has a (weak) inverse in $L^H(\mathcal{M}, \mathcal{W})$. When \mathcal{M} is further a simplicial model category, then for every pair (x, y) of objects $\text{Hom}_{L^H(\mathcal{M}, \mathcal{W})}(x, y)$ is naturally homotopy equivalent to the derived mapping space $\mathbb{R}\text{Hom}(x, y)$.

It follows that any model category \mathcal{M} gives functorially rise to the simplicial category $L^H(\mathcal{M}, \mathcal{W})$. Taking the nerve $N_\bullet(L^H(\mathcal{M}, \mathcal{W}))$ we obtain a simplicial space. Composing with the complete Segal Space replacement functor we get a functor $\mathcal{M} \rightarrow L_\infty(\mathcal{M}) := L_{\mathcal{CSeSp}}(N_\bullet(L^H(\mathcal{M}, \mathcal{W})))$ from model categories to $(\infty, 1)$ -categories (that is complete Segal spaces).

Example 1. Applying the above procedure to the model category of simplicial sets $s\text{Set}$, we obtain the $(\infty, 1)$ -category $s\text{Set}_\infty$. Similarly from the model category $CDGA$ of CDGAs we obtain the $(\infty, 1)$ -category $CDGA_\infty$. Note that a simplicial set is determined by its $(\infty, 0)$ path groupoid and therefore the category of simplicial sets should be thought of as the $(\infty, 1)$ category of all $(\infty, 0)$ groupoids. Further, the tensor product (over k) of algebras is a monoidal functor which gives $CDGA$ a structure of monoidal model category, see [19]. Thus $CDGA_\infty$ also inherits the structure of a symmetric monoidal $(\infty, 1)$ -category in the sense of [25, 29]. Similarly, the disjoint union of simplicial sets endows $s\text{Set}$ and $s\text{Set}_\infty$ with symmetric monoidal structures.

The model category of topological spaces yields the $(\infty, 1)$ -category Top_∞ . Since $s\text{Set}$ and Top are Quillen equivalent [15, 19], the associated $(\infty, 1)$ -categories are equivalent (as $(\infty, 1)$ -categories): $s\text{Set}_\infty \xrightarrow{\sim} Top_\infty$, where the left and right equivalences are respectively induced by the singular set and geometric realization functors.

One can also consider the pointed versions $s\text{Set}_{\infty*}$ and $Top_{\infty*}$ of the above $(\infty, 1)$ -categories (using the model categories of these pointed versions [19]).

Example 2. As recalled in Sect. 2.2, there are model categories $A\text{-Mod}$ and $A\text{-CDGA}$ of modules and commutative algebras over a CDGA A . Thus the above procedure gives us $(\infty, 1)$ -categories $A\text{-Mod}_\infty$ and $A\text{-CDGA}_\infty$ and the base change functor lifts to an $(\infty, 1)$ -functor. Further, if $f : A \rightarrow B$ is a weak equivalence, the natural functor $f_* : B\text{-Mod} \rightarrow A\text{-Mod}$ induces an equivalence $B\text{-Mod}_\infty \xrightarrow{\sim} A\text{-Mod}_\infty$ of $(\infty, 1)$ -categories since it is a Quillen equivalence.

Moreover, if $f : A \rightarrow B$ is a morphism of CDGAs, it induces a natural functor $f^* : A\text{-Mod} \rightarrow B\text{-Mod}$, $M \mapsto M \otimes_A B$, which is an equivalence of $(\infty, 1)$ -categories when f is a quasi-isomorphism, and is a (weak) inverse of f_* (see [30] or [20]). Here we also (abusively) denote $f^* : A\text{-Mod}_\infty \rightarrow B\text{-Mod}_\infty$ and $f_* : B\text{-Mod}_\infty \rightarrow A\text{-Mod}_\infty$ the (derived) functors of $(\infty, 1)$ -categories induced by f . Since we are working over a field of characteristic zero, the same results applies to monoids in $A\text{-Mod}$ and $B\text{-Mod}$, that is to the categories $A\text{-CDGA}_\infty$ and $B\text{-CDGA}_\infty$. Also note that, if $f : A \rightarrow B$, $g : A \rightarrow C$ are CDGAs homomorphisms, we can form the (homotopy) pushout $D \cong B \otimes_A^{\mathbb{L}} C$.

Example 3. We denote $E_n\text{-Alg}_\infty$ the $(\infty, 1)$ -category of E_n -algebras which is given by algebras over any E_n - (∞) -operads as introduced in [24, Section 5.1] in the symmetric monoidal $((\infty, 1)\text{-category}) (k\text{-Mod}_\infty, \otimes)$. It is equivalent to the $(\infty, 1)$ -category associated to model categories (deduced for instance from [16, Theorem 4.1.1]) of algebras over the usual operad of singular chains on the little n -dimensional disk operad or as algebras over the Barratt–Eccles operad (which is an Hopf operad) [4].

Remark 1. There are other functors that yields a complete Segal space out of a model category. For instance, one can use the classification diagram of Rezk [29] which yields an equivalent Segal space, see [6].

More generally, there are several model for $(\infty, 1)$ -categories and several equivalent ways to obtain an $(\infty, 1)$ -category out of a “homotopy theory”. We believe the results of this paper can easily be applied to the favorite model of the reader.

3. Derived Higher Hochschild Functor

3.1. Naive axiomatic approach to higher Hochschild homology. We first recall the standard construction of chain complexes computing higher Hochschild homology (also called Hochschild homology over spaces) following [14, 28]. The higher Hochschild complex is a functor $CH : sSet \times CDGA \rightarrow CDGA$. This functor is defined as follows: the tensor products $A \otimes B$ of two CDGAs has an natural structure of CDGA (in other words, $CDGA$ has a symmetric monoidal structure canonically induced by the underlying tensor product of chain complexes). Furthermore, the multiplication $A \otimes A \rightarrow A$ is an algebra homomorphism since A is commutative. It follows that a $CDGA$ can be thought of a strict symmetric monoidal functor from the category of finite sets with disjoint union to the category of chain complexes (whose value on a finite set J is given by $A^{\otimes J}$), which can be extended to the category of all sets by taking colimits. Given a simplicial set X_\bullet , thought of as a functor $\Delta^{op} \rightarrow sSet$, compose these two functors to obtain a simplicial complex $X_\bullet \mapsto A^{\otimes X_\bullet}$. The total complex (that is the geometric realization $|A^{\otimes X_\bullet}|$) of this simplicial complex is, by definition, $CH_{X_\bullet}^\bullet(A, A)$. In more details, we get the following explicit definitions.

Definition 2. First let $Y_\bullet : \Delta^{op} \rightarrow Sets_*$ be a finite pointed simplicial set, and for $k \geq 0$, we set $y_k := Y_k - \{*\}$ to be the complement of the base point in Y_k . Furthermore,

let $(A = \bigoplus_{i \in \mathbb{Z}} A^i, d, \bullet)$ be a differential graded, associative, commutative algebra, and $(M = \bigoplus_{i \in \mathbb{Z}} M^i, d_M)$ a differential graded module over A (viewed as a symmetric bimodule). Then, the **Hochschild chain complex of A with values in M over Y_\bullet** is defined as² $CH_{Y_\bullet}^n(A, M) := \bigoplus_{n \in \mathbb{Z}} CH_{Y_\bullet}^n(A, M)$, where

$$CH_{Y_\bullet}^n(A, M) := \bigoplus_{k \geq 0} (M \otimes A^{\otimes y_k})^{n+k}$$

is given by a sum of elements of total degree $n + k$. In order to define a differential D on $CH_{Y_\bullet}^n(A, M)$, we define morphisms $d_i : Y_k \rightarrow Y_{k-1}$, for $i = 0, \dots, k$ as follows. First note that for any map $f : Y_k \rightarrow Y_l$ of pointed sets, and for $m \otimes a_1 \otimes \dots \otimes a_{y_k} \in M \otimes A^{\otimes y_k}$, we denote by $f_* : M \otimes A^{\otimes y_k} \rightarrow M \otimes A^{\otimes y_l}$,

$$f_*(m \otimes a_1 \otimes \dots \otimes a_{y_k}) = (-1)^\epsilon n \otimes b_1 \otimes \dots \otimes b_{y_l}, \quad (1)$$

where $b_j = \prod_{i \in f^{-1}(j)} a_i$ (or $b_j = 1$ if $f^{-1}(j) = \emptyset$) for $j = 0, \dots, y_l$, and $n = m \bullet \prod_{i \in f^{-1}(\text{basepoint}), i \neq \text{basepoint}} a_i$. The sign ϵ in equation (1) is determined by the usual Koszul sign rule of $(-1)^{|x| \bullet |y|}$ whenever x moves across y . In particular, there are induced boundaries $(d_i)_* : CH_{Y_\bullet}^k(A, M) \rightarrow CH_{Y_\bullet}^{k-1}(A, M)$ and degeneracies $(s_j)_* : CH_{Y_\bullet}^k(A, M) \rightarrow CH_{Y_\bullet}^{k+1}(A, M)$, which we denote by abuse of notation again by d_i and s_j . Using these, the differential $D : CH_{Y_\bullet}^n(A, A) \rightarrow CH_{Y_\bullet}^n(A, A)$ is defined by letting $D(a_0 \otimes a_1 \otimes \dots \otimes a_{y_k})$ be equal to

$$\sum_{i=0}^{y_k} (-1)^{k+\epsilon_i} a_0 \otimes \dots \otimes d(a_i) \otimes \dots \otimes a_{y_k} + \sum_{i=0}^k (-1)^i d_i(a_0 \otimes \dots \otimes a_{y_k}),$$

where ϵ_i is again given by the Koszul sign rule, i.e., $(-1)^{\epsilon_i} = (-1)^{|a_0| + \dots + |a_{i-1}|}$. The simplicial conditions on d_i imply that $D^2 = 0$.

If $Y_\bullet : \Delta^{op} \rightarrow \text{Sets}$ is a finite (not necessarily pointed) simplicial set, we may still define $CH_{Y_\bullet}^n(A) := \bigoplus_{n \in \mathbb{Z}} CH_{Y_\bullet}^n(A, A)$ via the same formula as above, $CH_{Y_\bullet}^n(A, A) := \bigoplus_{k \geq 0} (A \otimes A^{\otimes y_k})_{n+k}$. Formula (1) again induces boundaries d_i and degeneracies s_i , which produce a differential D of square zero on $CH_{Y_\bullet}^n(A, A)$ as above.

If Y_\bullet is any simplicial set we define

$$CH_{Y_\bullet}^n(A, M) := \varinjlim_{\substack{K_\bullet \rightarrow Y_\bullet, \\ K_\bullet \text{ finite}}} CH_{K_\bullet}^n(A, M)$$

as the colimit over all finite simplicial sets. If Y_\bullet is finite, then this definition agrees with the previous ones thanks to the Yoneda lemma.

Remark 2. Note that due to our grading convention, if A is non graded, or concentrated in degree 0, then $HH_{Y_\bullet}^{\bullet}(A, A)$ is concentrated in non-positive degrees. In particular, our grading is opposite of the one in [21].

² We recall that we are using a cohomological type grading for our differential graded modules, see Convention (7) on page 7, hence the upper index n .

Note that the equation (1) also makes sense for any map of simplicial pointed sets $f : X_k \rightarrow Y_k$. Since A is graded commutative and M symmetric, $(f \circ g)_* = f_* \circ g_*$, hence $Y_\bullet \mapsto CH_{Y_\bullet}^\bullet(A, M)$ is a functor from the category of finite pointed simplicial sets to the category of simplicial k -vector spaces, see [28]. If $M = A$, $CH_{Y_\bullet}^\bullet(A)$ is a functor from the category of finite simplicial sets to the category of simplicial k -algebras.

Now note that any map $g : A \rightarrow B$ of CDGAs and any maps of modules $\rho : M \rightarrow N$ over $g : A \rightarrow B$, i.e. $\rho(am) = g(a)\rho(m)$, induces a map $CH_{Y_\bullet}^\bullet(g, \rho) : CH_{Y_\bullet}^\bullet(A, M) \rightarrow CH_{Y_\bullet}^\bullet(B, N)$ of simplicial vector spaces.

The chain complex $(CH_{Y_\bullet}^\bullet(A), D)$ inherits a structure of (differential graded) algebra. This is a formal consequence of the fact that $CH_{Y_\bullet}^\bullet(A)$ is a simplicial commutative algebra. Indeed, given two simplicial vector spaces V_\bullet and W_\bullet , one defines a simplicial structure on the simplicial space $(V \times W)_k := V_k \otimes W_k$ using the boundaries $d_i^V \otimes d_i^W$ and degeneracies $s_i^V \otimes s_i^W$. The *shuffle product* is (the collection of) maps $sh : V_p \otimes W_q \rightarrow (V \times W)_{p+q}$ defined by

$$sh(v \otimes w) = \sum_{(\mu, \nu)} sgn(\mu, \nu) (s_{\nu_q} \dots s_{\nu_1}(v) \otimes s_{\mu_p} \dots s_{\mu_1}(w)),$$

where (μ, ν) denotes a (p, q) -shuffle, i.e. a permutation of $\{0, \dots, p+q-1\}$ mapping $0 \leq j \leq p-1$ to μ_{j+1} and $p \leq j \leq p+q-1$ to ν_{j-p+1} , such that $\mu_1 < \dots < \mu_p$ and $\nu_1 < \dots < \nu_q$.

Since $CH_{Y_\bullet}^\bullet(A, M)$ is a simplicial vector space, we obtain an induced shuffle map $sh : CH_{Y_p}^\bullet(A, M) \otimes CH_{Y_q}^\bullet(B, N) \rightarrow CH_{Y_{p+q}}^\bullet(A \otimes B, M \otimes N)$ for any CDGAs A, B and modules M, N . Now, since A is a CDGA, the multiplication $\mu : A \otimes A \rightarrow A$ is an algebra map, and the map $\nu : M \otimes A \rightarrow M$ a map of A -modules. Composing these maps with the shuffle products we obtain the multiplication

$$sh_{Y_\bullet} : CH_{Y_\bullet}^\bullet(A, M) \otimes CH_{Y_\bullet}^\bullet(A) \xrightarrow{sh} CH_{Y_\bullet}^\bullet(A \otimes A, M \otimes A) \xrightarrow{CH_{Y_\bullet}^\bullet(\mu, \nu)} CH_{Y_\bullet}^\bullet(A, M).$$

Proposition 1. *The multiplication sh_{Y_\bullet} makes $CH_{Y_\bullet}^\bullet(A)$ a differential graded commutative algebra and $CH_{Y_\bullet}^\bullet(A, M)$ a DG-module over $CH_{Y_\bullet}^\bullet(A)$, which are natural in A and M .*

Proof. The proof of the algebra structure is given in [14, Proposition 2.4.2] and the proof of the module structure is the same.

Note that $CH_{pt_\bullet}^\bullet(A)$ is the (chain complex associated to the) constant simplicial CDGA A . In particular there is a canonical quasi-isomorphism $\eta : A = CH_{pt_0}^\bullet(A) \rightarrow CH_{pt_\bullet}^\bullet(A)$ splitting the augmentation map $CH_{pt_\bullet}^\bullet(A) \rightarrow CH_{pt_0}^\bullet(A)$. It follows from Proposition 1 above that if X_\bullet is a pointed simplicial set, the canonical map $pt_\bullet \rightarrow X_\bullet$ induces a natural A -module structure on $CH_{X_\bullet}^\bullet(A, M)$ (and an A -algebra structure on $CH_{X_\bullet}^\bullet(A)$). In other words, $CH_{X_\bullet}^\bullet(A, M)$ is naturally an A -module.

Summing up the previous discussion and proposition we obtain:

Corollary 1. *The rule $(Y_\bullet, A) \mapsto (CH_{Y_\bullet}^\bullet(A), D, sh_{Y_\bullet})$ is a functor $CH : sSet \times CDGA \rightarrow CDGA$. Similarly, the rule $(Y_\bullet, M) \mapsto (CH_{Y_\bullet}^\bullet(A, M), D, sh_{Y_\bullet})$ is a functor $CH : sSet_* \times A\text{-Mod} \rightarrow A\text{-Mod}$.*

Definition 3. The Hochschild homology of a CDGA A over a simplicial set X_\bullet is the cohomology³ $HH_{X_\bullet}^\bullet(A) = H^\bullet(CH_{X_\bullet}^\bullet(A), D)$ of the CDGA $(CH_{X_\bullet}^\bullet(A), D, sh)$ as a commutative graded algebra.

Further if X_\bullet is pointed and M is an A -module, the Hochschild homology of A with value in M over X_\bullet is the homology $HH_{X_\bullet}^\bullet(A, M) = H^*(CH_{X_\bullet}^\bullet(A, M), D)$ as a graded module over $HH_{X_\bullet}^\bullet(A)$.

Now let X be a topological space, we define the Hochschild homology of a CDGA A over X to be $HH_{S_\bullet(X)}^\bullet(A)$ where

$$S_\bullet(X) = Map(\Delta^\bullet, X)$$

is the singular simplicial set of X . If X is pointed, then $S_\bullet(X)$ is a pointed simplicial set and we define the Hochschild homology of a CDGA A with value in an A -module M over X to be $HH_{S_\bullet(X)}^\bullet(A, M)$.

The Hochschild chain functor satisfies the following properties which allows one to build *explicitly and easily* these chain complexes out of other simplicial sets and do computations (for instance, see [13, 14, 28]).

Proposition 2 (Tensor Products of CDGAs and disjoint union of simplicial sets). *Let A, B be two CDGAs. For any $X_\bullet \in sSet$, there is a canonical isomorphism*

$$CH_{X_\bullet}^\bullet(A \otimes B) \cong CH_{X_\bullet}^\bullet(A) \otimes CH_{X_\bullet}^\bullet(B)$$

of CDGAs. Further for any simplicial set Y_\bullet , one has a natural isomorphism

$$CH_{X_\bullet \amalg Y_\bullet}^\bullet(A) \cong CH_{X_\bullet}^\bullet(A) \otimes CH_{Y_\bullet}^\bullet(A)$$

of CDGAs and a natural isomorphism of modules

$$CH_{X_\bullet \amalg Y_\bullet}^\bullet(A, M) \cong CH_{X_\bullet}^\bullet(A, M) \otimes CH_{Y_\bullet}^\bullet(A)$$

if X_\bullet is a pointed simplicial set.

Proof. It follows from the canonical isomorphisms $(A \otimes B)^{\otimes n} \cong A^{\otimes n} \otimes B^{\otimes n}$ and $A^{\otimes n+m} \cong A^{\otimes n} \otimes A^{\otimes m}$.

Recall that, by functoriality, if $f : Y_\bullet \rightarrow X_\bullet$ is a map of simplicial sets, then for any CDGA A , we have a map of algebra $f_* : CH_{Y_\bullet}^\bullet(A) \rightarrow CH_{X_\bullet}^\bullet(A)$ which exhibits the Hochschild complex of A over X_\bullet as a module over the Hochschild complex of A over Y_\bullet . Let $Z_\bullet \rightarrow X_\bullet$, $Z_\bullet \rightarrow Y_\bullet$ be two maps of simplicial sets and let W_\bullet be a pushout $W_\bullet \cong X_\bullet \amalg_{Z_\bullet} Y_\bullet$.

Proposition 3. *There is a natural map of simplicial modules⁴*

$$CH_{X_\bullet}^\bullet(A, M) \otimes_{CH_{Z_\bullet}^\bullet(A, A)} CH_{Y_\bullet}^\bullet(A, A) \rightarrow CH_{W_\bullet}^\bullet(A, M)$$

which is a map of algebras if $M = A$ (with its natural module structure). If Z_\bullet injects into either $Z_\bullet \xrightarrow{f_\bullet} X_\bullet$ or $Z_\bullet \xrightarrow{g_\bullet} Y_\bullet$, then this map is in fact an isomorphism of $CH_{W_\bullet}^\bullet(A)$ -modules.

³ Recall that we are using a cohomological type grading for our differential graded modules, see Convention (7) on page 7.

⁴ The tensor product in Proposition 3 is the tensor product of (simplicial) modules over the simplicial differential graded commutative algebra $CH_{Z_\bullet}^\bullet(A, A)$. Passing to the Hochschild chain complexes, it induces a natural map of CDGAs and modules and yield a quasi-isomorphism if Z_\bullet injects into either X_\bullet or Y_\bullet , see [14, Corollary 2.4.3].

Proof. The proof [14, Lemma 2.1.6] given in the case $M = A$ applies to any module M .

Corollary 2. *The rule $(X_\bullet, A) \mapsto HH_{X_\bullet}^\bullet(A)$ is a functor $HH : sSet \times CDGA \rightarrow CGA$ which satisfies the following axioms*

1. **bimonoidal:** *Hochschild homology is monoidal with respect to the monoidal structures given by the disjoint union of simplicial sets and tensor products of algebras. In other words, there are natural isomorphisms:*

$$HH_{X_\bullet \times Y_\bullet}^\bullet(A) \cong HH_{X_\bullet}^\bullet(A) \otimes HH_{Y_\bullet}^\bullet(A), \quad HH_{X_\bullet}^\bullet(A \otimes B) \cong HH_{X_\bullet}^\bullet(A) \otimes HH_{X_\bullet}^\bullet(B).$$

2. **homotopy invariance :** *if $f : X_\bullet \rightarrow Y_\bullet$ and $g : A \rightarrow B$ are (weak) homotopy equivalences, then $HH(f, g) : HH_{X_\bullet}^\bullet(A) \rightarrow HH_{Y_\bullet}^\bullet(B)$ is an isomorphism.*
3. **point** *There is a natural isomorphism $HH_{pt}^\bullet(A) \cong A$*

A similar statement holds with the category of topological spaces instead of simplicial sets, and with the pointed analogs of these categories (as in Corollary 1).

Proof. This follows from Proposition 3, Proposition 2, Corollary 1 and Proposition 4 below.

The axioms listed in the above proposition are *not* enough to uniquely determine Hochschild homology as a functor. Indeed, we are missing an analog of the Excision/Mayer-Vietoris axioms in the classical Eilenberg–Steenrod axioms for singular homology. The analog of this axiom is similar to the *locality axiom* of a Topological Field Theory. In view of Proposition 3, we wish to compute the Hochschild homology over an union of two open sets as the tensor product of the Hochschild homology of each open tensored over the Hochschild homology over their intersection. This forces us to take *derived* tensor products. Thus a better framework for an axiomatic description of Hochschild chains is given by considering derived categories or $(\infty, 1)$ -categories. We deal with this *locality* axiom in Sect. 3.2 below. This axiom translates into an Eilenberg–Moore spectral sequence for Hochschild homology, see Corollary 3.

A crucial property of Hochschild chains which allows to pass to homotopy categories, is the fact, proved by Pirashvili [28], that the higher Hochschild chain complex is invariant along *quasi-isomorphisms* in both arguments.

Proposition 4 (Homotopy and homology invariance). *If $f : X_\bullet \rightarrow Y_\bullet$ is a map of simplicial sets inducing an isomorphism in homology $H_\bullet(X) \xrightarrow{\cong} H_\bullet(Y)$, then the map $CH_{X_\bullet}^\bullet(A, M) \rightarrow CH_{Y_\bullet}^\bullet(A, M)$ is a quasi-isomorphism.*

Further if $h : A \rightarrow B$ is a quasi-isomorphism of CDGAs, then the induced map $h_\bullet : CH_{X_\bullet}^\bullet(A) \rightarrow CH_{X_\bullet}^\bullet(B)$ is a quasi-isomorphism of CDGAs.

If Z_\bullet is a pointed simplicial set and M is a B -module, the induced map $h_\bullet : CH_{Z_\bullet}^\bullet(A, M) \rightarrow CH_{Z_\bullet}^\bullet(B, M)$ is a quasi-isomorphism of $CH_{Z_\bullet}^\bullet(A)$ -modules and if $\alpha : M \rightarrow N$ is a map of B -modules, the induced map $\alpha_\bullet : CH_{Z_\bullet}^\bullet(B, M) \rightarrow CH_{Z_\bullet}^\bullet(B, N)$ is a quasi-isomorphism of $CH_{Z_\bullet}^\bullet(B)$ -modules.

Proof. This is essentially due to Pirashvili [28]. Indeed, let Γ be the category of finite sets, then the Hochschild chain complex $CH_{X_\bullet}^\bullet(A)$ is isomorphic to the tensor product

$$k_{X_\bullet} \otimes_\Gamma \mathcal{L}(A)$$

of the left Γ -module $\mathcal{L}(A)$ and the simplicial right Γ -module k_{X_\bullet} . Here the left Γ -module $\mathcal{L}(A)$ is defined by $n \mapsto A^{\otimes n+1}$ and formula (1). The right Γ -module k_{X_\bullet} is defined by $n \mapsto \varinjlim_{K_\bullet \text{ finite}} k[\text{Hom}_\Gamma([n], K_\bullet)]$ where $[n]$ is the finite set $\{0, \dots, n\}$,

see [28]. A quasi-isomorphism of CDGAs induces a quasi-isomorphism of left Γ -module and similarly for a quasi-isomorphism of simplicial sets. Since each right Γ -module k_{X_m} ($m \in \mathbb{N}$) is a projective right Γ -module (see [28]), the tensor product $k_{X_\bullet} \otimes_\Gamma \mathcal{L}(A)$ is quasi-isomorphic to the derived tensor product $k_{X_\bullet} \otimes_\Gamma^{\mathbb{L}} \mathcal{L}(A)$. It follows that this complex is invariant along quasi-isomorphisms in both arguments (X_\bullet and A). The proof in the case of pointed simplicial sets and modules is the same with Γ replaced by the category of pointed finite sets.

3.2. Higher Hochschild as an $(\infty, 1)$ -functor. In this section we deal with axioms for the theory of higher Hochschild *chains* instead of mere homology. That is, we upgrade the previous section, in particular Corollary 2, to the setting of derived categories, or more precisely $(\infty, 1)$ -categories. In this setting, we will prove that the axioms determine *uniquely* the Hochschild chain as an $(\infty, 1)$ -functor (lifting Hochschild homology). These axioms are *not* specific to CDGA but rather come from the fact that any presentable $(\infty, 1)$ -category is (homotopically) canonically tensored over simplicial sets according to [22, Corollary 4.4.4.9].

Theorem 1. *There is a canonical equivalence $CH_{X_\bullet}(A) \cong X_\bullet \boxtimes A$ between the Hochschild chains and the tensor of A and X_\bullet , i.e. there are natural equivalences (in $sSet_\infty$)*

$$\text{Map}_{CDGA_\infty}(CH_{X_\bullet}(A), B) \cong \text{Map}_{sSet_\infty}(X_\bullet, \text{Map}_{CDGA_\infty}(A, B)). \quad (2)$$

In particular, the Hochschild chains lift as a functor of $(\infty, 1)$ -categories $CH : sSet_\infty \times CDGA_\infty \rightarrow CDGA_\infty$ which satisfies the following axioms

1. **value on a point:** *there is a natural equivalence $CH_{pt}^\bullet(A) \cong A$ of CDGAs.*
2. **monoidal:** *CH is monoidal with respect to both variables. Precisely, there are natural equivalences*

$$\begin{aligned} CH_{X_\bullet \amalg Y_\bullet}^\bullet(A) &\cong CH_{X_\bullet}^\bullet(A) \otimes CH_{Y_\bullet}^\bullet(A), \\ CH_{X_\bullet}^\bullet(A \otimes B) &\cong CH_{X_\bullet}^\bullet(A) \otimes CH_{X_\bullet}^\bullet(B). \end{aligned}$$

3. **homotopy gluing/pushout:** *CH sends homotopy pushout in $sSet_\infty$ to homotopy pushout in $CDGA_\infty$. More precisely, given maps $Z_\bullet \xrightarrow{f} X_\bullet$ and $Z_\bullet \xrightarrow{g} Y_\bullet$ in $sSet_\infty$, and $W_\bullet \cong X_\bullet \cup_{Z_\bullet}^h Y_\bullet$ a homotopy pushout, there is a natural equivalence*

$$CH_{W_\bullet}^\bullet(A) \cong CH_{X_\bullet}^\bullet(A) \otimes_{CH_{Z_\bullet}^\bullet(A)}^{\mathbb{L}} CH_{Y_\bullet}^\bullet(A).$$

Axiom (3) is the *locality* axiom (as for Topological Field Theories) playing the role of the excision/Mayer-Vietoris axiom for classical homology.

Note also that the relationship between Hochschild homology and tensoring a commutative algebra with the standard simplicial set model of the circle goes back to at least (McClure et al. [26]), where it was used in the context of spectra.

Proof. The second equivalence in the monoidal axiom follows from Proposition 2. The rest is an immediate consequence of [22, Corollary 4.4.4.9], Sect. 3.1 and the fact that the coproduct in $CDGA_\infty$ is given by the tensor product of CDGAs. Note that the axioms can also be proved easily and directly using the results of Sect. 3.1, namely Propositions 1, 2, 3 and 4.

Remark 3. As in Corollary 2, the (model) categories of simplicial sets and (compactly generated) topological spaces being Quillen equivalent, one can replace $sSet_\infty$ by its topological counterpart Top_∞ in Theorem 1, see Sect. 4.

The locality axiom (3) in Theorem 1 yields an Eilenberg–Moore type spectral sequence for computing higher Hochschild homology.

Corollary 3. *Given an homotopy pushout $W_\bullet \cong X_\bullet \cup_{Z_\bullet}^h Y_\bullet$, there is a natural strongly convergent spectral sequence of cohomological type of the form*

$$E_2^{p,q} := Tor_{p,q}^{HH_{Z_\bullet}^\bullet(A)}(HH_{X_\bullet}^\bullet(A), HH_{Y_\bullet}^\bullet(A)) \implies HH_{W_\bullet}^{p+q}(A)$$

where q is the internal grading. The spectral sequence is furthermore a spectral sequence of differential $HH_{Z_\bullet}^\bullet(A)$ -algebras.

Recall that we are considering a cohomological grading; in particular the spectral sequence is concentrated in the left half-plane with respect to this grading (and p is negative).

Proof. By Theorem 1, we have $CH_{W_\bullet}^\bullet(A) \cong CH_{X_\bullet}^\bullet(A) \otimes_{CH_{Z_\bullet}^\bullet(A)}^{\mathbb{L}} CH_{Y_\bullet}^\bullet(A)$ as CDGAs. Now the spectral sequence follows from standard results on derived tensor products (of $CH_{Z_\bullet}^\bullet(A)$ -modules) [20, Theorem 4.7]. That the spectral sequence is one of algebras comes from the fact, that we can choose a semi-free resolution of $CH_{X_\bullet}^\bullet$ by a $CH_{Z_\bullet}^\bullet$ -algebra.

Example 4. 1. It is a well-known fact, that the usual Hochschild complex of an associative algebra A $CH_\bullet(A) = CH_{S_1^\bullet}^\bullet(A)$ may be written as a Tor over the bimodule $A^e = A \otimes A^{op}$, see e.g. [21, Proposition 1.1.13]. More explicitly,

$$HH_\bullet(A) = Tor_\bullet^{A^e}(A, A).$$

Identifying $HH_\bullet^{pt^\bullet}(A) = A$, and $HH_{\{pt_\bullet, pt_\bullet^-\}}^\bullet(A) = A \otimes A^{op} = A^e$, where pt_\bullet^- denotes the point with opposite orientation, we see that, in this case, the spectral sequence of Corollary 3 collapses at the E_2 level,

$$Tor^{HH_{\{pt_\bullet, pt_\bullet^-\}}^\bullet(A)}(HH_{pt_\bullet}^\bullet(A), HH_{pt_\bullet}^\bullet(A)) = HH_{S_1^\bullet}^\bullet(A)$$

where we used that $S_1^1 \cong pt_\bullet \cup_{\{pt_\bullet, pt_\bullet^-\}}^h pt_\bullet$ and further that when A is commutative, $A^{op} = A$.

2. Let $f_\bullet : Z_\bullet \rightarrow X_\bullet$ be a map of simplicial spaces. Then, the mapping cone $(C_f)_\bullet$ is given as the homotopy pushout $(C_f)_\bullet \cong X_\bullet \cup_{Z_\bullet}^h pt_\bullet$. We obtain a spectral sequence computing $HH_{(C_f)_\bullet}^\bullet(A)$,

$$Tor_{p,q}^{HH_{Z_\bullet}^\bullet(A)}(HH_{X_\bullet}^\bullet(A), A) \implies HH_{(C_f)_\bullet}^\bullet(A)$$

3. In a more straightforward way, we may use the gluing property to give explicit models for spaces with cubic subdivisions. Our starting point is given by models for interval I_\bullet and the square I_\bullet^2 via,

$$CH_{I_\bullet}^\bullet(A) = \sum_k A^{\otimes(k+2)} \quad \text{and} \quad CH_{I_\bullet^2}^\bullet(A) = \sum_k A^{\otimes(k+2)^2}.$$

The i^{th} differential is given in the case of I_\bullet by multiplying the i^{th} and $(i+1)^{\text{th}}$ tensor factors, and in the case of I_\bullet^2 by multiplying the i^{th} and $(i+1)^{\text{th}}$ column of tensor factors and the i^{th} and $(i+1)^{\text{th}}$ rows of tensor factors simultaneously. For more detail, see [14, Example 2.3.4]. Then we obtain the Hochschild complex for the cylinder $C_\bullet = I_\bullet^2 \cup_{(I_\bullet \cup I_\bullet)} I_\bullet$ by gluing I_\bullet^2 and I_\bullet along $I_\bullet \cup I_\bullet$ on opposite edges of I_\bullet^2 , and with this the torus $T_\bullet = C_\bullet \cup_{(I_\bullet \cup I_\bullet)} I_\bullet$ by gluing the remaining sides together. A more elaborate version of this is given in [14, Example 2.3.2]. By using similar, but more elaborate considerations, one can in fact obtain Hochschild models over any surface, see [14, Section 3.1].

Higher Hochschild chain complexes behaves much like cochains of mapping spaces (see [14, sections 2.2, 2.4]). Indeed they satisfy a kind of exponential law:

Proposition 5 (Finite Products of simplicial sets). *Let X_\bullet, Y_\bullet be simplicial sets. Then there is a natural equivalence (in $CDGA_\infty$)*

$$CH_{X_\bullet \times Y_\bullet}^\bullet(A) \xrightarrow{\sim} CH_{X_\bullet}^\bullet(CH_{Y_\bullet}^\bullet(A)).$$

Proof. It follows from Corollary 2.4.4 of [14].

The Hochschild chain functor $CH : sSet_\infty \times CDGA_\infty \rightarrow CDGA_\infty$ is essentially (up to equivalences) determined by the homotopy pushout axiom, coproduct and its value on a point. In other words, it is the *unique* $(\infty, 1)$ -functor (up to natural equivalences) satisfying the three axioms (value on a point, coproduct and locality) listed below in Theorem 2. This is once again a consequence of the fact that higher Hochschild is a tensor. The precise uniqueness statement is :

Theorem 2 (Derived Uniqueness). *Let $(X_\bullet, A) \mapsto F_{X_\bullet}(A)$ be a bifunctor $sSet_\infty \times CDGA_\infty \rightarrow CDGA_\infty$ which satisfies the following three axioms.*

1. **value on a point:** *There is a natural equivalence of CDGAs $F_{pt_\bullet}(A) \cong A$.*
2. **coproduct:** *There are natural equivalences*

$$F_{\coprod_I (X_i)_\bullet}(A) \cong \varinjlim_{\substack{K \subset I \\ K \text{ finite}}} \bigotimes_{k \in K} F_{(X_k)_\bullet}(A)$$

3. **homotopy gluing/pushout:** *F sends homotopy pushout in $sSet_\infty$ to homotopy pushout in $CDGA_\infty$. More precisely, given two maps $Z_\bullet \xrightarrow{f} X_\bullet$ and $Z_\bullet \xrightarrow{g} Y_\bullet$ in $sSet_\infty$, and $W_\bullet \cong X_\bullet \bigcup_{Z_\bullet}^h Y_\bullet$ a homotopy pushout, one has a natural equivalence*

$$F_{W_\bullet}(A) \cong F_{X_\bullet}(A) \otimes_{F_{Z_\bullet}(A)}^{\mathbb{L}} F_{Y_\bullet}(A).$$

Then F is naturally equivalent to the higher Hochschild chains bifunctor CH as a bifunctor i.e. as an object in $Hom_{(\infty, 1)\text{-cat}}(sSet_\infty \times CDGA_\infty, CDGA_\infty)$.

Proof. This is a consequence of the fact that $CDGA_\infty$ (and actually any presentable $(\infty, 1)$ -category) is uniquely tensored over $sSet_\infty$, see [22, Corollary 4.4.4.9]. Alternatively it can be proved by noticing that any simplicial set X_\bullet is a (homotopy) colimit of its skeletal filtration $sk_n X_\bullet$, which in turns is obtained by taking a homotopy pushout of $sk_{n-1} X_\bullet$ with coproducts of standard model Δ^n_\bullet of the simplices which are contractible. This (using the axioms) implies an equivalence $F_{sk_n X_\bullet}(A) \xrightarrow{\sim} CH^\bullet_{sk_n X_\bullet}(A)$ which commutes with the inclusions $sk_{n-1} X_\bullet \hookrightarrow sk_n X_\bullet$. Thus by axiom (3), there are natural equivalences

$$F_{X_\bullet}(A) \cong F_{\coprod_{n \in \mathbb{N}} sk_n X_\bullet}(A) \xrightarrow[\otimes]{\mathbb{L}} F_{\coprod_{n \in \mathbb{N}} sk_n X_\bullet}(A) \quad (3)$$

and similarly for Hochschild chains by Theorem 1 so that the conclusion follows.

Remark 4. If A is concentrated in non-positive degrees, then $CH^\bullet_{X_\bullet}(A)$ is also concentrated in non-positive degrees. This happens for instance if A is the CDGA associated to a simplicial (non-graded) commutative algebra. In that case, it is possible to replace $CDGA$ and $CDGA_\infty$ in Theorem 1 and Theorem 2 by $CDGA^{\leq 0}$ the category of CDGAs concentrated in non-positive degrees and $CDGA_\infty^{\leq 0}$ its associated $(\infty, 1)$ -categories (the proofs being unchanged).

Remark 5. Again, one can replace $sSet_\infty$ by its topological counterpart Top_∞ in Theorem 2, see Proposition 8.

Remark 6. Note that one can deduce from the coproduct axiom (2) in Theorem 2 and the natural equivalence (3) that the natural map $\varinjlim F_{sk_n X_\bullet}(A) \xrightarrow{\sim} F_{X_\bullet}(A)$ is an equivalence. This is in particular true for Hochschild chains:

$$\varinjlim_{n \geq 0} CH_{sk_n X_\bullet}(A) \xrightarrow{\sim} CH_{X_\bullet}(A). \quad (4)$$

Remark 7. If $G : CDGA_\infty \rightarrow CDGA_\infty$ is a functor, one can replace the value on a point axiom by the existence of a natural quasi-isomorphism $F_{pt}(A) \cong G(A)$. The proof of the Theorem 2 shows the following

Corollary 4. *Let $G : CDGA_\infty \rightarrow CDGA_\infty$ be a functor and $(X_\bullet, A) \mapsto F_{X_\bullet}(A)$ be a bifunctor $sSet_\infty \times CDGA_\infty \rightarrow CDGA_\infty$ which satisfies the axioms (2) and (3) in Theorem 2 and with axiom (1) replaced by $F_{pt_\bullet}(A) \cong G(A)$. Then $F_{X_\bullet}(A)$ is naturally equivalent $CH^\bullet_{X_\bullet}(G(A))$.*

For instance, consider the bifunctor given by $(X_\bullet, A) \mapsto CH^\bullet_{X_\bullet}(A) \otimes CH^\bullet_{X_\bullet}(B)$ whose value on a point is the functor $A \mapsto A \otimes B$. By Corollary 4, this functor is isomorphic to $(X, A) \mapsto CH^\bullet_{X_\bullet}(A \otimes B)$ which gives another proof of the fact that the Hochschild chains preserve finite coproduct of CDGAs.

Corollary 5. *The Hochschild chain bifunctor $CH : sSet_\infty \times CDGA_\infty \rightarrow CDGA_\infty$ commutes with finite colimits in $sSet_\infty$ and all colimits in $CDGA_\infty$, that is there are natural equivalences*

$$\begin{aligned} CH \varinjlim_{\mathcal{F}} X_{i_\bullet}(A) &\cong \varinjlim_{\mathcal{F}} CH_{X_{i_\bullet}}(A) \quad (\text{for a finite category } \mathcal{F}), \\ CH_{X_\bullet}(\varinjlim A_i) &\cong \varinjlim CH_{X_\bullet}(A_i). \end{aligned}$$

Proof. Any finite colimits can be obtained by a composition of coproducts and pushouts (or coequalizers). Thus the result for colimits in $sSet_\infty$ follows from Theorem 1, Axioms (2) and (3). Let $\varinjlim_{i \in \mathcal{I}} A_i$ be a non-empty colimit of CDGAs and let i_0 be an object

in the indexing category \mathcal{I} . By functoriality we can define a functor $G_{\mathcal{I}} : CDGA_\infty \rightarrow CDGA_\infty$ by the formula $A \mapsto G_{\mathcal{I}}(A) := \varinjlim_{i \in \mathcal{I}} \tilde{B}_i$ where $\tilde{B}_i \cong A_i$ if $i \neq i_0$ and $\tilde{B}_{i_0} \cong A$. In other words we fix all the variables but the one indexed by i_0 . Now applying Corollary 4 to the bifunctor defined by $F_{X_\bullet}(B) \cong \varinjlim CH_{X_\bullet}^\bullet(\tilde{B}_i)$ we get a natural equivalence

$$CH_{X_\bullet}^\bullet(\varinjlim_{i \in \mathcal{I}} A_i) \cong \varinjlim_{i \in \mathcal{I}} CH_{X_\bullet}^\bullet(A_i).$$

Since the simplicial module $n \mapsto CH_{X_n}^\bullet(k)$ is isomorphic to the constant simplicial k -algebra $n \mapsto k$, the result also follows for empty colimits.

Example 5. By Corollary 5, given two maps $f : R \rightarrow A$ and $g : R \rightarrow B$ of CDGAs, there is a natural equivalence (in $CDGA_\infty$)

$$CH_{X_\bullet}^\bullet\left(A \otimes_R B\right) \cong CH_{X_\bullet}^\bullet(A) \otimes_{CH_{X_\bullet}^\bullet(R)} CH_{X_\bullet}^\bullet(B).$$

3.3. Pointed simplicial sets and modules. In this section we quickly explain how to add an A -module M to the story developed in Sect. 3.2.

Let A be a CDGA and recall from Example 2 the $(\infty, 1)$ -category $A\text{-Mod}_\infty$ induced by the (model category) $A\text{-Mod}$ of A -modules. Similarly, the model category of pointed simplicial sets yields the $(\infty, 1)$ -category $sSet_{\infty*}$ of pointed simplicial sets (Example 1).

Since the inclusion $pt_\bullet \rightarrow X_\bullet$ is always a cofibration, the canonical equivalence $CH_{X_\bullet}^\bullet(A, M) \cong M \otimes_A CH_{X_\bullet}^\bullet(A)$ given by Proposition 3 implies that

$$CH_{X_\bullet}^\bullet(A, M) \cong M \otimes_A CH_{X_\bullet}^\bullet(A) \cong M \otimes_{CH_{pt_\bullet}^\bullet(A)} CH_{X_\bullet}^\bullet(A). \quad (5)$$

naturally in $A\text{-Mod}_\infty$ and $CH_{X_\bullet}^\bullet(A)\text{-Mod}_\infty$.

Proposition 2, Proposition 3, Proposition 5, Theorem 1 and its proof imply

Theorem 3. *The Hochschild chain lifts as a bifunctor of $(\infty, 1)$ -categories $CH_{(-)}(A, -) : sSet_{\infty*} \times A\text{-Mod}_\infty \rightarrow A\text{-Mod}_\infty$ which satisfies the following axioms*

1. **value on a point:** $CH_{pt_\bullet}^\bullet(A, M) \cong M$ in $A\text{-Mod}_\infty$.
2. **action of CH :** $CH_{X_\bullet}^\bullet(A, M)$ is naturally a $CH_{X_\bullet}^\bullet(A)$ -module, i.e. the Hochschild chain lifts as an $(\infty, 1)$ -functor $CH_{X_\bullet}^\bullet(A, -) : A\text{-Mod}_\infty \rightarrow CH_{X_\bullet}^\bullet(A)\text{-Mod}_\infty$.
3. **bimonoidal:** there is a natural equivalence

$$CH_{X_\bullet \amalg Y_\bullet}^\bullet(A, M) \cong CH_{X_\bullet}^\bullet(A, M) \otimes CH_{Y_\bullet}^\bullet(A)$$

in $A\text{-Mod}_\infty$ as well as in $CH_{X_\bullet \amalg Y_\bullet}^\bullet(A)\text{-Mod}_\infty$. For $M \in A\text{-Mod}_\infty$ and $N \in B\text{-Mod}_\infty$, there is a natural equivalence

$$CH_{X_\bullet}^\bullet(A \otimes B, M \otimes N) \cong CH_{X_\bullet}^\bullet(A, M) \otimes CH_{X_\bullet}^\bullet(B, N)$$

in $A \otimes B\text{-Mod}_\infty$ and $CH_{X_\bullet}^\bullet(A \otimes B)\text{-Mod}_\infty$.

4. **locality:** let $f : Z_{\bullet} \rightarrow X_{\bullet}$ and $g : Z_{\bullet} \rightarrow Y_{\bullet}$ be maps in $sSet_{\infty*}$. There is a natural equivalence

$$CH_{X_{\bullet} \cup_{Z_{\bullet}} Y_{\bullet}}^{\bullet}(A, M \otimes_A^{\mathbb{L}} N) \cong CH_{X_{\bullet}}^{\bullet}(A, M) \otimes_{CH_{Z_{\bullet}}^{\bullet}(A)}^{\mathbb{L}} CH_{Y_{\bullet}}^{\bullet}(A, N)$$

in $A\text{-Mod}_{\infty}$ and $CH_{X_{\bullet} \cup_{Z_{\bullet}} Y_{\bullet}}^{\bullet}(A)\text{-Mod}_{\infty}$. If $N = A$, then only X_{\bullet} needs to be pointed and the maps may be in $sSet_{\infty}$.

5. **product:** There is a natural equivalence

$$CH_{X_{\bullet} \times Y_{\bullet}}^{\bullet}(A, M) \xrightarrow{\sim} CH_{X_{\bullet}}^{\bullet}(CH_{Y_{\bullet}}^{\bullet}(A), CH_{Y_{\bullet}}^{\bullet}(A, M))$$

in $A\text{-Mod}_{\infty}$ and $CH_{X_{\bullet} \times Y_{\bullet}}^{\bullet}(A)\text{-Mod}_{\infty}$.

Proposition 6. Let $G : A\text{-Mod}_{\infty} \rightarrow A\text{-Mod}_{\infty}$ be an $(\infty, 1)$ -functor and let $\mathcal{M} : sSet_{\infty*} \times A\text{-Mod}_{\infty} \rightarrow A\text{-Mod}_{\infty}$ be any $(\infty, 1)$ -bifunctor which satisfies the following axioms

- i) **value on a point:** $\mathcal{M}(pt_{\bullet}, M) \cong G(M)$ naturally in $A\text{-Mod}_{\infty}$.
- ii) **action of CH :** $\mathcal{M}(X_{\bullet}, M)$ is naturally a $CH_{X_{\bullet}}^{\bullet}(A)$ -module.
- iii) **locality:** There is a natural equivalence (in $A\text{-Mod}_{\infty}$)

$$\mathcal{M}(X_{\bullet} \cup_{Z_{\bullet}}^h Y_{\bullet}, M) \cong \mathcal{M}(X_{\bullet}, M) \otimes_{CH_{Z_{\bullet}}^{\bullet}(A)}^{\mathbb{L}} CH_{Y_{\bullet}}^{\bullet}(A)$$

Then \mathcal{M} is naturally equivalent to $CH_{(-)}(A, G(-))$ as an $(\infty, 1)$ -bifunctor.

Note that Axiom ii) is needed to make sense of Axiom iii).

Proof. Let Y_{\bullet} be in $sSet_{*}$. The locality axiom for the pushout $pt_{\bullet} \leftarrow pt_{\bullet} \xrightarrow{g} Y_{\bullet}$ (where we take $X_{\bullet} = pt_{\bullet}$) gives natural equivalences

$$\mathcal{M}(X_{\bullet}, M) \cong \mathcal{M}(pt_{\bullet}, M) \otimes_{CH_{pt_{\bullet}}^{\bullet}(A)}^{\mathbb{L}} CH_{X_{\bullet}}^{\bullet}(A) \cong G(M) \otimes_{CH_{pt_{\bullet}}^{\bullet}(A)}^{\mathbb{L}} CH_{X_{\bullet}}^{\bullet}(A)$$

where the last equivalence follows from Axiom i).

Theorem 3 yields a relative version of the Eilenberg–Moore spectral sequence. In fact the proof of Corollary 3 (using Theorem 3 instead of Theorem 1) yields

Corollary 6. Given an homotopy pushout $W_{\bullet} \cong X_{\bullet} \cup_{Z_{\bullet}}^h Y_{\bullet}$, there is a natural strongly convergent spectral sequence of cohomological type of the form

$$E_2^{p,q} := Tor_{p,q}^{HH_{Z_{\bullet}}^{\bullet}(A)}(HH_{X_{\bullet}}^{\bullet}(A, M), HH_{Y_{\bullet}}^{\bullet}(A, N)) \implies HH_{W_{\bullet}}^{p+q}\left(A, M \otimes_A^{\mathbb{L}} N\right)$$

where q is the internal grading. The spectral sequence is furthermore a spectral sequence of differential $HH_{Z_{\bullet}}^{\bullet}(A)$ -modules.

Recall that we are considering a cohomological grading; thus the spectral sequence lies in the left half-plane with respect to this grading (and p is negative).

Example 6. If X_{\bullet} , Y_{\bullet} and Z_{\bullet} are contractible, the spectral sequence in Corollary 6 boils down to the usual (see [20]) Eilenberg–Moore spectral sequence $Tor_{p,q}^{H^{\bullet}(A)}(H^{\bullet}(M), H^{\bullet}(N)) \implies H^{p+q}(M \otimes_A^{\mathbb{L}} N)$ of differential $H^{\bullet}(A)$ -modules.

Remark 8. Besides the Eilenberg Moore spectral sequence 6, there is also an Atiyah–Hirzebruch kind of spectral sequence for higher Hochschild chains: the skeletal filtration of a simplicial set X_\bullet induces a decreasing filtration $\cdots \supset F^p \cdots \supset F^{-1} \supset F^0 \supset \{0\}$ of $CH_{X_\bullet}^\bullet(A, M)$, where $F^p := \bigoplus_{n \leq -p} CH_{X_n}^\bullet(A, M)$. This filtration yields a left half-plane spectral sequence of cohomological type with exiting differential and further, the cohomology of the associated graded $\bigoplus_p F^p / F^{p+1}$ is the Hochschild chain complex over X_\bullet of the CGA $H^\bullet(A)$ with value in $H^\bullet(M)$. Hence we get from [7]:

Proposition 7. *There is a strongly convergent spectral sequence of cohomological type*

$$E_{p,q}^2 := HH_{X_\bullet}^{p+q}(H^\bullet(A), H^\bullet(M))^q \implies HH_{X_\bullet}^{p+q}(A, M)$$

where q is the internal degree. If $M = A$, this is a spectral sequence of CDGAs.

For the sake of completeness, we also mention that there is another spectral sequence to compute higher Hochschild due to Pirashvili, which is the Grothendieck spectral sequence associated to the composition of functors $X_\bullet \mapsto k_{X_\bullet} \mapsto k_{X_\bullet} \otimes_{\mathbb{L}_F} \mathcal{L}(A)$ that was defined in Proposition 4. See [28, Theorem 2.4] for details.

4. Factorization Algebras and Derived Hochschild Functor Over Spaces

The main goal of this section is to prove that Hochschild chains are a special kind of *factorization algebras* in the sense of [8] (allowing to compute it using covers or CW-decomposition).

4.1. The Hochschild $(\infty, 1)$ -functor in Top . The Quillen equivalence between the model categories of simplicial sets and topological spaces induces an equivalence $Top_\infty \xrightarrow{S_\infty} sSet_\infty$ of $(\infty, 1)$ -categories (Example 1). Here S_∞ is the $(\infty, 1)$ -functor lifting the singular set functor $X \mapsto S_\bullet(X) = Map(\Delta^\bullet, X)$. Recall that to any space X we naturally associate the CDGA $CH_X^\bullet(A) = CH_{S_\bullet(X)}^\bullet(A)$, the Hochschild chains of A over X . The canonical adjunction map $X_\bullet \rightarrow S_\bullet(|X_\bullet|)$ yields a natural quasi-isomorphism $CH_{X_\bullet}^\bullet(A) \rightarrow CH_{|X_\bullet|}^\bullet(A)$ of CDGAs by Proposition 4.

From the above equivalence $Top_\infty \xrightarrow{\sim} sSet_\infty$ (or alternatively by changing $sSet_\infty$ to Top_∞ in all the proofs in Sects. 3.2 and 3.3), we deduce the following topological counterpart to the results of Sects. 3.2 and 3.3.

Proposition 8. i) *The Hochschild chain over spaces functor $(X, A) \mapsto CH_X^\bullet(A)$ lifts as an $(\infty, 1)$ -bifunctor $CH : Top_\infty \times CDGA_\infty \rightarrow CDGA_\infty$ fitting into the commutative diagram*

$$\begin{array}{ccc} sSet_\infty \times CDGA_\infty & \xrightarrow{CH} & CDGA_\infty \\ \uparrow S_\infty \simeq & \nearrow CH & \\ Top_\infty \times CDGA_\infty & & \end{array}$$

that satisfies all the axioms of Theorem 1 (with Top_∞ instead of $sSet_\infty$).

ii) *Further, up to natural equivalences of $(\infty, 1)$ -bifunctors, it is the only bifunctor $Top_\infty \times CDGA_\infty \rightarrow CDGA_\infty$ satisfying the axioms of Theorem 2 (with Top_∞ instead of $sSet_\infty$).*

iii) Replacing $sSet_\infty$ by Top_∞ and $sSet_{\infty*}$ by $Top_{\infty*}$, the analogs of Corollary 3, Proposition 5, Corollary 5, Theorem 3 and Corollary 6 hold.

The fact that Hochschild chains are computed by taking colimits over finite simplicial sets has the following translation for topological spaces.

Proposition 9 (Compact support). *Let X be (weakly homotopic to) a CW -complex, A a $CDGA$ and M an A -module. There are natural equivalences*

$$\lim_{\substack{K \rightarrow X \\ K \text{ compact}}} (CH_K^\bullet(A)) \xrightarrow{\sim} CH_X^\bullet(A) \quad \text{and} \quad \lim_{\substack{K \rightarrow X \\ K \text{ compact}}} (CH_K^\bullet(A, M)) \xrightarrow{\sim} CH_X^\bullet(A, M)$$

in $CDGA_\infty$ and $CH_X^\bullet(A)\text{-Mod}_\infty$ respectively.

Proof. Since $CH_X^\bullet(A, M) \cong M \otimes_A^L CH_X^\bullet(A)$ in $CH_X^\bullet(A)\text{-Mod}_\infty$, we only need to prove that the first map is an equivalence.

We first assume X to be a CW -complex of finite dimension n . Let X_\bullet be a simplicial set model of X with no non-degenerate simplices in dimension $m > n$. Then, given a finite simplicial set K_\bullet , any map $f : K_\bullet \rightarrow X_\bullet$ factors through a finite simplicial set \tilde{K}_\bullet with no non-degenerate simplices in dimension $m > n$. Thus, the realization $|\tilde{K}_\bullet|$ is compact. Conversely, if \tilde{K} is a compact subset of the CW -complex X , it has a simplicial model K_\bullet with finitely many non-degenerate simplices. Further, any map $K \rightarrow X$ has a compact image, since X is Hausdorff, and thus factors through a compact subset of X . We get a zigzag

$$\begin{aligned} \lim_{\substack{K \rightarrow X \\ K \text{ compact}}} CH_K^\bullet(A) &\xleftarrow{\sim} \lim_{\substack{\tilde{K} \subset X \\ \tilde{K} \text{ compact}}} CH_{\tilde{K}}^\bullet(A) \xrightarrow{\sim} \lim_{\tilde{K}_\bullet \in \text{FNDS}(X_\bullet)} CH_{\tilde{K}_\bullet}^\bullet(A) \\ &\xrightarrow{\sim} \lim_{\substack{K_\bullet \rightarrow X_\bullet \\ K_\bullet \text{ finite}}} CH_{K_\bullet}^\bullet(A) \xrightarrow{\sim} CH_{X_\bullet}^\bullet(A) \xrightarrow{\sim} CH_X^\bullet(A) \end{aligned}$$

where the first and third arrows are equivalences since they are induced by cofinal functors. Here $\text{FNDS}(X_\bullet)$ is the set of simplicial subsets of X_\bullet with finitely many non-degenerate simplices. That the other arrows in the zigzag are equivalences follows from Proposition 8 and thus the result is proved for finite dimensional CW -complexes.

We now reduce the general case to the finite dimensional one. Let X_\bullet be a simplicial set model of X . The geometric realization $|sk_n X_\bullet|$ of $sk_n X_\bullet$ is a finite dimensional CW -complex, and, if K is compact, any map $f : K \rightarrow X$ factors as the composition $K \rightarrow |sk_n X_\bullet| \hookrightarrow X$ for some n . Hence the natural map

$$\lim_n \left(\lim_{\substack{\tilde{K} \rightarrow |sk_n X_\bullet| \\ \tilde{K} \text{ compact}}} CH_{\tilde{K}}^\bullet(A) \right) \longrightarrow \lim_{\substack{K \rightarrow X \\ K \text{ compact}}} CH_K^\bullet(A)$$

is a natural equivalence in $CDGA_\infty$. The result now follows from the finite dimensional case, the natural equivalence $\lim_n CH_{sk_n X_\bullet}^\bullet(A) \xrightarrow{\sim} CH_{X_\bullet}^\bullet(A)$ (see Remark 6), and Proposition 8.i).

Lemma 1. *Let X_0 be (weakly homotopic to) a CW-complex and X be (weakly homotopic to) a CW-complex obtained from X_0 by attaching a countable family $(C_n)_{n \in \mathbb{N}}$ of cells. We let X_n be the result of attaching the first n cells. For any CDGA A , one has a natural equivalence*

$$\varinjlim_{n \in \mathbb{N}} CH_{X_n}^\bullet(A) \xrightarrow{\sim} CH_X^\bullet(A)$$

in $CDGA_\infty$, as well as $\varinjlim_{n \in \mathbb{N}} CH_{X_n}^\bullet(A, M) \xrightarrow{\sim} CH_X^\bullet(A)$ in $CH_X^\bullet(A)\text{-Mod}_\infty$.

Proof. Since $CH_X^\bullet(A, M) \cong M \otimes_A^L CH_X^\bullet(A)$ as $CH_X^\bullet(A)$ -modules, we only need to consider the case of $CH_X^\bullet(A)$. Further, the cells C_i are homeomorphic to euclidean balls and the attaching maps have domain given by their boundaries. Thus we may assume that each X_n is obtained from a simplicial set model $(X_0)_\bullet$ of X_0 by adding finitely many non-degenerate simplices. Thus we get a sequence of cofibrations of simplicial sets (i.e. degree wise injective maps) $(X_0)_\bullet \hookrightarrow (X_1)_\bullet \cdots \hookrightarrow (X_n)_\bullet \cdots \hookrightarrow X_\bullet = \varinjlim_{n \in \mathbb{N}} (X_n)_\bullet$ which are

(homotopy) models for the sequence of maps $X_0 \rightarrow X_1 \rightarrow \cdots \rightarrow X$. By definition of Hochschild chains, there is a canonical equivalence $\varinjlim_{\substack{K_\bullet \rightarrow X_\bullet \\ K_\bullet \text{ finite}}} CH_{K_\bullet}^\bullet(A) \cong CH_{X_\bullet}^\bullet(A)$ of

CDGAs. The maps $(X_n)_\bullet \rightarrow X_\bullet$ assemble to give a map of colimits

$$\varinjlim_{n \in \mathbb{N}} \left(\varinjlim_{\substack{K_\bullet \rightarrow (X_n)_\bullet \\ K_\bullet \text{ finite}}} CH_{K_\bullet}^\bullet(A) \right) \longrightarrow \varinjlim_{\substack{K_\bullet \rightarrow X_\bullet \\ K_\bullet \text{ finite}}} CH_{K_\bullet}^\bullet(A). \quad (6)$$

Given a finite set K_i and a map $f_i : K_i \rightarrow X_i$, the image $f_i(K_i)$ lies in some $(X_n)_i$ since $X_\bullet = \varinjlim_{n \in \mathbb{N}} (X_n)_\bullet$ and $f_i(K_i)$ is finite. This proves that the family $K_\bullet \rightarrow (X_n)_\bullet$ of maps from a pointed set into some $(X_n)_\bullet$ is cofinal and thus the map (6) is an equivalence in $CDGA_\infty$.

Remark 9. It is possible to enhance the result of Lemma 1 in the following way. One can take any space X_\emptyset and a space X obtained by attaching a family $(C_i)_{i \in I}$ of other spaces to it. Then, essentially the same argument as the one in Lemma 1 shows that $CH_X^\bullet(A)$ is the colimit $\varinjlim CH_{X_F}^\bullet(A)$ over all possible subspaces $X_F \subset X$ obtained by attaching finitely many C_i 's.

Let us conclude this section by giving an analog of Leray acyclic cover theorem/Mayer Vietoris principle for Hochschild chains.

Let X be a topological space and $\mathcal{U} = (U_i)_{i \in I}$ be a *good cover* for X , i.e. a cover such that the U_i and all of their nonempty finite intersections are contractible. We denote $N_\bullet(I)$ the nerve of the cover, that is $N_0(I) = I$, $N_1(I)$ is the set of pairs of indices i_0, i_1 such that $U_{i_0} \cap U_{i_1} \neq \emptyset$ and so on.

Corollary 7. *Let X be a topological space and $\mathcal{U} = (U_i)_{i \in I}$ be a good cover for X such that the inclusions $U_i \cap U_j \rightarrow U_i$ are cofibrations. Then there is a natural equivalence*

$$CH_X^\bullet(A) \xrightarrow{\sim} A^{\otimes I} \bigotimes_{A^{\otimes N_1(I)}}^{\mathbb{L}} A^{\otimes I}$$

in $CDGA_\infty$. Here the left and right module structure are induced by the two canonical projections $N_1(I) \rightarrow N_0(I) = I$ given by $(i, j) \mapsto i$, $(i, j) \mapsto j$.

Proof. Since each U_i is contractible, the natural map $CH_{U_i}^\bullet(A) \rightarrow CH_{pt}^\bullet(A) \cong A$ is an equivalence by Theorem 1 and similarly (when $U_i \cap U_j$ is not empty) for the natural map $CH_{U_i \cap U_j}^\bullet(A) \rightarrow A$ because \mathcal{U} is a good cover. Since X is the coequalizer $\coprod_{N_1(I)} U_i \cap U_j \rightrightarrows \coprod_I U_i \rightarrow X$, the result follows from the coproduct axiom (2) and the gluing axiom (3) in Theorem 1 (or Proposition 8).

4.2. (Pre)Factorization algebras. We now explain a relationship between factorization algebras (as defined by Costello and Gwilliam [8, 9]) and Higher Hochschild chains.

Let A be a CDGA and X be a topological space. We denote $Op(X)$ the set of open subsets of X . For every open subset V of X and a family of disjoint open subsets $U_1, \dots, U_n \subset V$, there is a canonical morphism of CDGAs

$$\mu_{U_1, \dots, U_n, V} : CH_{U_1}^\bullet(A) \otimes \cdots \otimes CH_{U_n}^\bullet(A) \rightarrow (CH_V^\bullet(A))^{\otimes n} \rightarrow CH_V^\bullet(A)$$

induced by functoriality by the inclusions $U_i \hookrightarrow V$ and the multiplication in $CH_V^\bullet(A)$.

These maps are the structure maps of a prefactorization algebra on X in the sense of [8, 9]. Note that for a (possibly $(\infty, 1)$ -) symmetric monoidal category (\mathcal{C}, \otimes) , we denote by $PFac_X(\mathcal{C})$ the $((\infty, 1)$ -) category of **prefactorization algebras on X taking values in \mathcal{C}** (see [8]). In particular, $PFac_X(CDGA)$ is the category of *commutative* prefactorization algebras on X as defined in [8]. We will actually be only interested in the case where \mathcal{C} is an $((\infty, 1)$ -)category of algebras over an (∞) -Hopf operad. We have:

Lemma 2. *The rule $U \mapsto CH_U^\bullet(A)$ together with the maps $\mu_{U_1, \dots, U_n, V}$ define a natural structure of a prefactorization algebra on X . Further*

1. *The above rule $A \mapsto \left((CH_{(U)}^\bullet(A))_{U \in Op(X)}; (\mu_{U_1, \dots, U_n, V}) \right)$ defines a functor $\mathcal{CH}_X : CDGA \rightarrow PFac_X(CDGA)$.*
2. *\mathcal{CH}_X lifts as an $(\infty, 1)$ -functor $\mathcal{CH}_X : CDGA_\infty \rightarrow PFac_X(CDGA_\infty)$.*

Proof. First we note that $CH_\emptyset^\bullet(A) \cong k$ and that the maps $\mu_{U_1, \dots, U_n, V}$ are CDGAs morphisms. Further, if V_1, \dots, V_l is a collection of pairwise disjoint open subsets of $V \in Op(X)$ and U_1, \dots, U_n is another family of pairwise disjoint open subsets of V such that each U_i is contained in some V_j , we can form the diagram

$$\begin{array}{ccc} \bigotimes_{j=1}^l \left(\bigotimes_{U_i \subset V_j} CH_{U_i}^\bullet(A) \right) & \xrightarrow{\mu_{U_1, \dots, U_n, V}} & CH_V^\bullet(A) \\ \downarrow \bigotimes_{j=1}^l \left(\mu_{(U_i \subset V_j), V_j} \right) & \nearrow \mu_{V_1, \dots, V_l, V} & \\ \bigotimes_{j=1}^l (CH_{V_j}^\bullet(A)) & & \end{array}$$

which is commutative by functoriality of Hochschild chains. This proves that the rule $U \mapsto CH_U^\bullet(A)$ is a prefactorization algebra with value in the category $CDGA$. The naturality follows from the naturality of Hochschild chains in the algebra variable and the lift to the $(\infty, 1)$ -framework follows from (the proof of) Proposition 8 and Theorem 1.

In particular, the prefactorization algebra $U \mapsto CH_U^\bullet(A)$ is a *commutative* prefactorization algebra.

Following the terminology of [8,9], we said that an open cover \mathcal{U} of $U \in Op(X)$ is **factorizing** if, for all finite collections x_1, \dots, x_n of distinct points in U , there are *pairwise disjoint* open subsets U_1, \dots, U_k in \mathcal{U} such that $\{x_1, \dots, x_n\} \subset \bigcup_{i=1}^k U_i$. To define a factorization algebra, we need to introduce the Čech complex of a prefactorization algebra \mathcal{F} . Let \mathcal{U} be a cover and denote $P\mathcal{U}$ the set of finite pairwise disjoint open subsets $\{U_1, \dots, U_n, U_i \in \mathcal{U}\}$. Now the **Čech complex** $\check{C}(\mathcal{U}, \mathcal{F})$ is the chain (bi-)complex

$$\check{C}(\mathcal{U}, \mathcal{F}) = \bigoplus_{P\mathcal{U}} \mathcal{F}(U_1) \otimes \dots \otimes \mathcal{F}(U_n) \leftarrow \bigoplus_{P\mathcal{U} \times P\mathcal{U}} \mathcal{F}(U_1 \cap V_1) \otimes \dots \otimes \mathcal{F}(U_n \cap V_m) \leftarrow \dots$$

where the horizontal arrows are induced by the alternate sum of the natural inclusions as for the usual Čech complex of a cosheaf (see [8]). Let us introduce a convenient notation for the Čech complex: given $\alpha_1, \dots, \alpha_k \in P\mathcal{U}$, we denote

$$\mathcal{F}(\alpha_1, \dots, \alpha_k) = \bigotimes_{U_{i_1} \in \alpha_1, \dots, U_{i_k} \in \alpha_k} \mathcal{F}(U_{i_1} \cap \dots \cap U_{i_k}),$$

The prefactorization algebra structure yields, for all $j = 1, \dots, k$, natural maps $\mathcal{F}(\alpha_1, \dots, \alpha_k) \rightarrow \mathcal{F}(\alpha_1, \dots, \tilde{\alpha}_j, \dots, \alpha_k)$. The Čech complex of \mathcal{F} can be simply written as

$$\check{C}(\mathcal{U}, \mathcal{F}) = \bigoplus_{k>0} \bigoplus_{\alpha_1, \dots, \alpha_k \in P\mathcal{U}} \mathcal{F}(\alpha_1, \dots, \alpha_k)[k-1]. \quad (7)$$

The prefactorization algebra structure also induce a canonical map $\check{C}(\mathcal{U}, \mathcal{F}) \rightarrow \mathcal{F}(U)$.

A prefactorization algebra \mathcal{F} on X (with value in $CDGA$ or $k\text{-Mod}$) is said to be a **factorization algebra** if, for all open subset $U \in Op(X)$ and every factorizing cover \mathcal{U} of U , the canonical map

$$\check{C}(\mathcal{U}, \mathcal{F}) \rightarrow \mathcal{F}(U)$$

is a quasi-isomorphism (see [8,9]). When \mathcal{F} is a commutative factorization algebra, the sequence $\bigoplus_{\alpha_1, \dots, \alpha_k \in P\mathcal{U}} \mathcal{F}(\alpha_1, \dots, \alpha_k)[k-1]$ is naturally a simplicial CDGA and thus the Čech complex $\check{C}(\mathcal{U}, \mathcal{F})$ has a natural structure of CDGA.

Note that X itself is always a factorizing cover. A Hausdorff space usually admits many different factorizing covers. This is in particular true for manifolds. Indeed, choosing a Riemannian metric on a manifold X yields a nice factorizing cover given by the set of geodesically convex neighborhoods of every point in X .

It is shown in [8] that, if \mathcal{U} is a *basis* for the topology of a space X which is also a *factorizing cover*, and \mathcal{F} is a \mathcal{U} -factorization algebra, then one obtains a factorization algebra $i_*^{\mathcal{U}}(\mathcal{F})$ on X defined by

$$i_*^{\mathcal{U}}(\mathcal{F})(V) := \check{C}(\mathcal{U}_V, \mathcal{F}) \quad (8)$$

where \mathcal{U}_V is the cover of V consisting of open subsets in \mathcal{U} which are also subsets of V . We recall that a \mathcal{U} -**factorization algebra** is like a factorization algebra, except that $\mathcal{F}(U)$ is only defined for $U \in \mathcal{U}$ and further that we only require a quasi-isomorphism $\check{C}(\mathcal{V}, \mathcal{F}) \xrightarrow{\sim} \mathcal{F}(U)$ for factorizing covers \mathcal{V} of U consisting of open sets in \mathcal{U} .

A **factorizing good cover** is a good cover which is also a factorizing cover. For instance, any CW-complex has a factorizing good cover. Admitting a basis of factorizing good cover is a sufficient condition to prove that the Hochschild prefactorization algebra \mathcal{CH}_X is a factorization algebra:

Theorem 4. *Let X be a topological space with a factorizing good cover and A be a CDGA. Assume further that there is a basis of open sets in X which is also a factorizing good cover. The prefactorization algebra $\mathcal{CH}_X : U \mapsto CH_U^\bullet(A)$ given by Lemma 2 is a factorization algebra on X .*

In particular, for any factorizing cover \mathcal{U} of X , there is a canonical equivalence of CDGAs

$$CH_X^\bullet(A) \cong \check{C}(\mathcal{U}, \mathcal{CH}_X)$$

For instance the theorem applies to all manifolds (that we always assume to be paracompact) and more generally to CW-complexes.

Proof. Let \mathcal{U} be a factorizing good cover. We first prove that the rule $U \mapsto CH_U^\bullet(A)$ is a \mathcal{U} -factorization algebra. Since, we already know that $U \mapsto CH_U^\bullet(A)$ is a prefactorization algebra (Lemma 2), we only need to prove that, for any $U \in \mathcal{U}$ and any factorizing cover \mathcal{V} of U consisting of open sets in \mathcal{U} , the canonical map $\check{C}(\mathcal{V}, \mathcal{CH}_X) \xrightarrow{\sim} CH_U^\bullet(A)$ is a quasi-isomorphism. Let us denote $P\mathcal{V}$ the set of finite pairwise disjoint open subsets $\{(U_1, \dots, U_n, U_i \in \mathcal{V})\}$. Now the Čech complex $\check{C}(\mathcal{V}, \mathcal{CH}_X)$ is the chain (bi-)complex

$$\bigoplus_{P\mathcal{V}} CH_{U_1}^\bullet(A) \otimes \dots \otimes CH_{U_n}^\bullet(A) \leftarrow \bigoplus_{P\mathcal{V} \times P\mathcal{V}} CH_{U_1 \cap V_1}^\bullet(A) \otimes \dots \otimes CH_{U_n \cap V_m}^\bullet(A) \leftarrow \dots$$

where the horizontal arrows are induced by the alternate sum of the natural inclusions (see [8]). Since \mathcal{U} is a good cover, Theorem 1 and the prefactorization algebra structure of \mathcal{CH}_X gives a natural equivalence of chain complexes

$$\begin{array}{ccc} \bigoplus_{P\mathcal{V}} CH_{U_1}^\bullet(A) \otimes \dots \otimes CH_{U_n}^\bullet(A) & \longleftarrow & \bigoplus_{P\mathcal{V} \times P\mathcal{V}} \left(\bigotimes CH_{U_i \cap V_j}^\bullet(A) \right) \longleftarrow \dots \\ \downarrow \simeq & & \downarrow \simeq \\ \bigoplus_{P\mathcal{V}} A \otimes \dots \otimes A & \longleftarrow & \bigoplus_{P\mathcal{V} \times P\mathcal{V}} A \otimes \dots \otimes A \longleftarrow \dots \end{array} \quad (9)$$

We can form a simplicial set $N_\bullet(\mathcal{V})$ given by the nerve of the cover \mathcal{V} . Since \mathcal{V} is factorizing, the canonical map

$$\varinjlim_{\substack{K_\bullet \xrightarrow{\text{disj}} N_\bullet(\mathcal{V}) \\ K_\bullet \text{ finite}}} CH_{K_\bullet}^\bullet(A) \longrightarrow \varinjlim_{\substack{K_\bullet \rightarrow N_\bullet(\mathcal{V}) \\ K_\bullet \text{ finite}}} CH_{K_\bullet}^\bullet(A) \cong CH_{N_\bullet(\mathcal{V})}(A) \quad (10)$$

(where the left colimit is over maps whose images are required to be disjoint open subsets) is an equivalence. The bottom line of diagram (9) now identifies with the left colimit of the map (10), hence with Hochschild chain complex of nerve $N_\bullet(\mathcal{V})$. Since \mathcal{U} is a good cover, by the Nerve Theorem (or Leray acyclic cover), the geometric realization of $N_\bullet(\mathcal{V})$ is quasi-isomorphic to the reunion $\bigcup_{\mathcal{V}} U_i = U$. Since U is assumed to be

contractible, we get from above and Proposition 4 a natural equivalence (of prefactorization algebras) $\check{C}(\mathcal{V}, \mathcal{CH}_X)(U) \xrightarrow{\sim} CH_U^\bullet(A) (\cong A)$. Thus the \mathcal{U} -prefactorization algebra $U \mapsto CH_U^\bullet(A)$ is a \mathcal{U} -factorization algebra. We denote $CH_{\mathcal{U}}$ this \mathcal{U} -factorization algebra.

To conclude, we are left to prove that the induced factorization algebra $i_*^{\mathcal{U}}(CH_{\mathcal{U}})$ on X is equivalent to \mathcal{CH}_X as a prefactorization algebra. Let V be an open subset of X . By [8], there is a natural equivalence $i_*^{\mathcal{U}}(CH_{\mathcal{U}})(V) \cong \check{C}(\mathcal{U}_V, CH_{\mathcal{U}})$ (where \mathcal{U}_V is the cover of V consisting of open subsets in \mathcal{U} which are also subsets of V). Since the cover \mathcal{U}_V is a good cover, as in the case where V was in \mathcal{U} above, there is a natural equivalence

$$\begin{array}{ccc} \bigoplus_{P\mathcal{U}_V} CH_{U_1}^\bullet(A) \otimes \cdots \otimes CH_{U_n}^\bullet(A) & \longleftarrow & \bigoplus_{P\mathcal{U}_V \times P\mathcal{U}_V} \left(\bigotimes CH_{U_i \cap V_j}^\bullet(A) \right) \longleftarrow \cdots \\ \downarrow \cong & & \downarrow \cong \\ \bigoplus_{P\mathcal{U}_V} A \otimes \cdots \otimes A & \longleftarrow & \bigoplus_{P\mathcal{U}_V \times P\mathcal{U}_V} A \otimes \cdots \otimes A \longleftarrow \cdots \end{array}$$

where, as for diagram (9) above, the bottom line is equivalent to the Hochschild chain complex $CH_{N_\bullet(\mathcal{U}_V)}^\bullet(A)$ of the simplicial set $N_\bullet(\mathcal{U}_V)$ given by the nerve of \mathcal{U}_V . Since \mathcal{U} is a good cover, we can again use the Nerve Theorem to see that $CH_{N_\bullet(\mathcal{U}_V)}^\bullet(A) \cong CH_{\bigcup_{U \in \mathcal{U}_V} U}^\bullet(A) \cong CH_V^\bullet(A)$ and thus to get the natural equivalence (of prefactorization algebras) $\check{C}(\mathcal{V}, i_*^{\mathcal{U}}(CH_{\mathcal{U}}))(U) \xrightarrow{\sim} CH_V^\bullet(A) \cong \mathcal{CH}_X(V)$.

If \mathcal{F} is a factorization algebra on X (with value in $k\text{-Mod}$ or $k\text{-Mod}_\infty$), and $f : X \rightarrow Y$ is a continuous map, one can define the **pushforward** $f_*(\mathcal{F})$ by the formula $f_*(\mathcal{F})(V) = \mathcal{F}(f^{-1}(V))$ which actually is a factorization algebra on Y , see [8]. Costello and Gwilliam [8, Section 3.a] have defined the **factorization homology** $HF(\mathcal{F})$ of \mathcal{F} as the pushforward $p_*(\mathcal{F})$ where $p : X \rightarrow pt$ is the unique map. In other words, we have natural equivalences

$$HF(\mathcal{F}) \cong p_*(\mathcal{F}) \cong \mathcal{F}(X) \cong \check{C}(\mathcal{U}, \mathcal{F}) \quad (11)$$

in $k\text{-Mod}_\infty$ (for any factorizing cover \mathcal{U} of X). Note that, despite its name, $HF(\mathcal{F})$ is a cochain complex (up to equivalences) and, in particular a *derived* object (which may be thought as the “derived global sections” of \mathcal{F}). If \mathcal{F} has value in $CDGA$, then $HF(\mathcal{F})$ is an object in $CDGA_\infty$ too. Theorem 4 and Lemma 2 yields

Corollary 8. *Let X be a CW-complex. The Hochschild prefactorization functor \mathcal{CH}_X is actually a functor $\mathcal{CH}_X : CDGA_\infty \rightarrow \text{Fac}_X(CDGA_\infty)$. Further, the factorization homology of $\mathcal{CH}_X(A)$ is equivalent to $CH_X^\bullet(A)$ (as an object of $CDGA_\infty$); in other words the following diagram commutes:*

$$\begin{array}{ccc} CDGA_\infty & \xrightarrow{CH_X^\bullet(-)} & CDGA_\infty \\ \mathcal{CH}_X \downarrow & \nearrow HF(-) & \\ \text{Fac}_X(CDGA_\infty) & & \end{array}$$

For manifolds, we will give below another geometric interpretation of the functor \mathcal{CH}_X in terms of embeddings of manifolds in euclidean spaces (see Example 7, Corollary 9 and Remark 11).

A factorization algebra \mathcal{F} on a manifold is said to be **locally constant**, if, the natural map $\mathcal{F}(U) \rightarrow \mathcal{F}(V)$ is a quasi-isomorphism when $U \subset V$ are homeomorphic to a ball (see [8, 9]). Furthermore, we call \mathcal{F} a **commutative constant factorization algebra** (on X), if there is a CDGA A , and natural quasi-isomorphisms $\mathcal{F}(U) \rightarrow A$ for any open $U \subset X$ homeomorphic to a ball. Here, natural means that for any pairwise disjoint open subsets homeomorphic to a ball $U_1, \dots, U_n \in V$ of a contractible open subset $V \in X$ also homeomorphic to a ball, the following diagram is commutative in $k\text{-Mod}_\infty$

$$\begin{array}{ccc} \mathcal{F}(U_1) \otimes \dots \otimes \mathcal{F}(U_n) & \xrightarrow{\mu_{U_1, \dots, U_n, V}} & \mathcal{F}(V) \\ \downarrow & & \downarrow \\ A^{\otimes n} & \xrightarrow{m^{(n)}} & A \end{array} \quad (12)$$

where $m^{(n)}$ is the $(n - 1)$ -times iterated multiplication of A .

Example 7. A class of examples of locally constant factorization algebras occurs as follows. By results of Lurie [24] (also see Proposition 10 below), the data of a locally constant factorization algebra on \mathbb{R}^n is the same as the data of an E_n -algebra. Thus any CDGA yields a locally constant factorization algebra (denoted \mathcal{A} for the moment) on \mathbb{R}^n (for any $n \geq 1$) and also, by restriction, on any open subset of \mathbb{R}^n .

Now, let X be any manifold, and let $i : X \hookrightarrow \mathbb{R}^n$ be an embedding of X in \mathbb{R}^n . Let NX be an open tubular neighborhood of X in \mathbb{R}^n . We write $p : NX \rightarrow X$ for the bundle map. Any factorization algebra \mathcal{F} on \mathbb{R}^n restricts to a factorization algebra $\mathcal{F}|_{NX}$ on NX and the pushforward $p_*(\mathcal{F}|_{NX})$ is a factorization algebra on X . Thus, a CDGA A yields a locally constant factorization algebra $p_*(\mathcal{A}|_{NX})$ on X for any manifold X . Since $p_*(\mathcal{F}|_{NX})(U) \cong \mathcal{F}(p^{-1}(U))$, it is easy to check that $p_*(\mathcal{A}|_{NX})$ is indeed locally constant. Since by construction there is a natural quasi-isomorphism $\mathcal{A}(B) \xrightarrow{\sim} A = \mathcal{A}(\mathbb{R}^n)$ for any open ball $B \subset \mathbb{R}^n$, the induced locally constant factorization algebra $p_*(\mathcal{A}|_{NX})$ satisfies that there exists a natural equivalence $\mathcal{A}|_{NX}(B) \xrightarrow{\sim} A$ for any open set $B \in \text{Op}(X)$ homeomorphic to a ball. In other words, we can think to the factorization algebra $p_*(\mathcal{A}|_{NX})$ as being *constant* (and commutative).

Note that the previous analysis can be extended to any space X which embeds as the base of locally trivial fibration $U \rightarrow X$ where U is an open in some \mathbb{R}^n and the fibers are homeomorphic to a ball.

Example 8. Let G be a discrete group acting properly discontinuously on a manifold X . According to [8], any G -equivariant factorization algebra \mathcal{F} on X yields a factorization algebra \mathcal{F}^G on X/G . Further, it is easy to check that, if \mathcal{F} is locally constant then so is \mathcal{F}^G too. In particular any CDGA A yields a locally constant factorization algebra on \mathbb{R}^n/G for any discrete group acting properly discontinuously on \mathbb{R}^n .

The next corollary describes what *constant* commutative factorization algebra are; namely they all are equivalent to derived Hochschild chains for some CDGA.

Corollary 9. *Let X be a manifold and \mathcal{F} a commutative factorization algebra such that there exists a CDGA A and a natural equivalence $\mathcal{F}(B) \xrightarrow{\sim} A$ for any open set*

$B \in \text{Op}(X)$ homeomorphic to a ball. Then \mathcal{F} is equivalent to the Hochschild chain factorization algebra $\mathcal{CH}_X(A)$ (given by Lemma 2).

In particular, there is a natural equivalence $HF(\mathcal{F}) \cong CH_X^\bullet(A)$.

Proof. Choosing a Riemannian metric on X , the set $\text{Ball}^g(X)$ of geodesic open balls is factorizing. Further, \mathcal{F} being constant, we have, for any $\alpha = \{B_1, \dots, B_k\} \in \text{Ball}^g(X)$, natural equivalences

$$\mathcal{F}(\alpha) = \mathcal{F}(B_1) \otimes \dots \otimes \mathcal{F}(B_k) \cong \bigotimes_{i=1}^k A \cong CH_{B_1}^\bullet(A) \otimes \dots \otimes CH_{B_k}^\bullet(A) \cong \mathcal{CH}_X(\alpha)$$

Similarly, there are natural equivalences

$$\mathcal{F}(\alpha_1, \dots, \alpha_j) \cong \mathcal{CH}_X(\alpha_1, \dots, \alpha_j)$$

for any $\alpha_1, \dots, \alpha_j \in P\text{Ball}^g(X)$. Thus, by Theorem 4, we have

$$\begin{aligned} HF(\mathcal{F}) &\cong \check{C}(\text{Ball}^g(X), \mathcal{F}) \\ &\cong \bigoplus_{k>0} \bigoplus_{\alpha_1, \dots, \alpha_k \in P\text{Ball}^g(X)} \mathcal{F}(\alpha_1, \dots, \alpha_k)[k-1] \\ &\cong \bigoplus_{k>0} \bigoplus_{\alpha_1, \dots, \alpha_k \in P\text{Ball}^g(X)} \mathcal{CH}_X(\alpha_1, \dots, \alpha_k)[k-1] \cong CH_X^\bullet(A). \end{aligned}$$

It follows that we have an equivalence $\check{C}(\text{Ball}^g(X), \mathcal{F}) \cong \check{C}(\text{Ball}^g(X), \mathcal{CH}_X)$. The same analysis can be made for any open subset $U \in \text{Op}(X)$ instead of X and the naturality of Hochschild chains ensures that the equivalence $\mathcal{F}(U) \cong CH_U^\bullet(A)$ is natural in U .

Remark 10. Note that the above Corollary 9 can be extended to manifolds with corners, where, by a locally constant factorization algebra on a manifold with corners, we mean a factorization algebra which is locally constant if, whenever restricted to the strata (which are manifolds), it is locally constant see [2, 8]. One can extend the definition of constant factorization algebra in the same way.

Let us also sketch how Corollary 9 can be used in the general case of *locally constant commutative factorization algebras*. Let \mathcal{A} be a locally constant factorization algebra on a manifold M and assume that there is a codimension 1 submanifold (possibly with corners) N of M with a trivialization $N \times I$ of its neighborhood such that M is decomposable as $M = X \cup_{N \times I} Y$ where X, Y are submanifolds (with corners) of M glued along $N \times I$. The inclusion $i : N \times I \rightarrow X$ induces a map of factorization algebras $i_*(\mathcal{A}|_{N \times I}) \rightarrow \mathcal{A}|_X$, which gives a structure of $\mathcal{A}|_{N \times I}$ -module to $\mathcal{A}|_X$ since \mathcal{A} is commutative.

Lemma 3. *If $\mathcal{A}|_Y$ is constant (say $\mathcal{A}|_Y(B) \cong A$ for a CDGAA and any ball $B \in \text{Op}(Y)$), \mathcal{A} is equivalent to $\mathcal{A}|_X \bigotimes_{\mathcal{CH}_{N \times I}(A)}^{\mathbb{L}} \mathcal{CH}_Y(A)$ in $\text{Fac}_M(CDGA_\infty)$ (where the factorization algebras are pushforward to M along the natural inclusions). In particular,*

$$HF(\mathcal{A}) \cong HF(\mathcal{A}|_X) \bigotimes_{CH_{N \times I}^\bullet(A)}^{\mathbb{L}} CH_Y^\bullet(A).$$

Using a handle decomposition of M , one can use the lemma above to compute the homology of a locally constant commutative factorization algebra \mathcal{A} in terms of (iterated) derived tensor products of derived Hochschild functors (see Sect. 5.2 for a related construction).

Proof. It follows from Corollary 10 and Lemma 5.

Remark 11. Corollary 9 implies that the factorization algebras $p_*(\mathcal{A}|_{N_X})$ are independent (up to equivalences in $\text{Fac}_X(k\text{-Mod}_\infty)$) of the choices of the embedding and of the tubular neighborhood made in Example 7. Indeed, if $i_1 : X \hookrightarrow \mathbb{R}^{n_1}$ and $i_2 : X \hookrightarrow \mathbb{R}^{n_2}$ are two embeddings of a manifold X in an euclidean space, and given two choices $\mathbb{R}^{n_1} \supset N_1 X \xrightarrow{p_1} X$, $\mathbb{R}^{n_2} \supset N_2 X \xrightarrow{p_2} X$ of tubular neighborhoods, Corollary 9 implies that there are natural equivalences

$$p_{1*}(\mathcal{A}|_{N_1 X})(U) \xrightarrow{\sim} CH_\bullet^U(A) \xleftarrow{\sim} p_{2*}(\mathcal{A}|_{N_2 X})(U)$$

for open subsets $U \subset X$.

Example 9. Let M be a manifold and $A = C^\infty(M)$ its algebra of functions. Also let Σ^g be a compact Riemann surface of genus g and \mathcal{F} the factorization algebra (see Theorem 4) on Σ^g defined by the rule $U \mapsto C_U^\bullet(C^\infty(M))$ (here, in the definition of Hochschild chains, the tensor product over k is the *completed* tensor product so that $C^\infty(M) \otimes C^\infty(M) \cong C^\infty(M \times M)$). Let $\mathcal{O}^n(M)$ denote the de Rham n -forms on M , viewed as complex concentrated in degree 0.

An analogue of Hochschild–Kostant–Rosenberg theorem for Hochschild chains over surfaces [14, Theorem 4.3.3] implies that the factorization homology of \mathcal{F} on Σ^g is given by

$$HF(\mathcal{F}) \cong S_{C^\infty(M)}\left(\Omega^1(M)[2]\right) \otimes_{C^\infty(M)} S_{C^\infty(M)}\left(\Omega^1(M)\right) \otimes_{C^\infty(M)} \bigotimes_{i=1 \dots 2g} \Omega^\bullet(M)[\bullet]$$

where $V[n]$ is the graded space $(V[n])^i = V^{i+n}$ (i.e. with cohomological degree shifted down by n) and $S_{C^\infty(M)}(W)$ is the symmetric graded algebra of a graded $C^\infty(M)$ -module W . In terms of graded-geometry the above isomorphism is equivalent to saying that $HF(\mathcal{F})$ is (equivalent to) the algebra of functions of the graded manifold

$$HF(\mathcal{F}) \cong T[2](M) \oplus \bigoplus_{i=1}^{2g} T[1]M.$$

We now study a homotopical strengthening of the locally constant condition. We said that a factorization algebra on X is **strongly locally constant** if the natural map $\mathcal{F}(U) \rightarrow \mathcal{F}(V)$ is a quasi-isomorphism when $U \subset V$ are contractibles. Let A be a CDGA; we say that a factorization algebra \mathcal{F} is **strongly constant of type A** , if there exists a natural quasi-isomorphism $\mathcal{F}(U) \rightarrow A$ for any contractible U . Here, natural means, that for any pairwise disjoint contractible open subsets $U_1, \dots, U_n \in V$ of a contractible open subset $V \in X$, the following diagram (similar to diagram 12) is commutative in $k\text{-Mod}_\infty$

$$\begin{array}{ccc}
 \mathcal{F}(U_1) \otimes \cdots \otimes \mathcal{F}(U_n) & \xrightarrow{\mu_{U_1, \dots, U_n, V}} & \mathcal{F}(V) \\
 \downarrow & & \downarrow \\
 A^{\otimes n} & \xrightarrow{m^{(n)}} & A
 \end{array}$$

Example 10. The factorization algebras given by Theorem 4 are strongly constant of type A .

Example 11. Let X be a manifold, A a CDGA and $p_*(\mathcal{A}_{|NX})$ be the factorization algebra on X constructed in Example 7. Corollary 9 implies that $p_*(\mathcal{A}_{|NX})$ is a *strongly constant* factorization algebra.

Now let X be a topological space that embeds as a retract $i : X \xrightarrow{\sim} UX : r$, where UX is an open subset of some \mathbb{R}^n (where n can be infinite). Then, similarly to the manifold case, the CDGA A yields a factorization algebra \mathcal{A} on \mathbb{R}^n and a factorization algebra $r_*(\mathcal{A}_{|UX})$ on X . The above paragraph implies that \mathcal{A} is strongly constant. Further if the fibers of r are contractible, then $r_*(\mathcal{A}_{|UX})$ is *strongly constant of type A* .

We have the following analogue of Corollary 9 for strongly constant factorization algebras on a topological space that admits a factorizing good cover (for instance those given by Example 11 when X is a CW-complex)

Corollary 10. Let X be a topological space with a factorizing good cover and A be a CDGA. Let \mathcal{F} be a factorization algebra on X .

1. Assume \mathcal{F} is strongly constant of type A , then one has a natural equivalence $H\mathcal{F}(\mathcal{F}) \cong CH_X^\bullet(A)$ in $k\text{-Mod}_\infty$.
2. Assume that there is a basis \mathcal{B} of open sets which is a factorizing good cover and that \mathcal{F} is a factorization algebra which satisfies the strongly constant condition (of type A) with respect to opens in \mathcal{B} , i.e., there exists a natural quasi-isomorphism $\mathcal{F}(U) \rightarrow A$ for $U \in \mathcal{B}$ (which is automatically contractible). Then, there is a natural equivalence of factorization algebras $\mathcal{F} \cong \mathcal{CH}_X(A)$ between \mathcal{F} and the Hochschild prefactorization algebra given by Lemma 2 (in particular, \mathcal{F} is strongly constant of type A).

Proof. The first assertion is proved as in Corollary 9 using a factorizing good cover \mathcal{U} for X instead of $\text{Ball}^g(X)$ (and using a proof similar to the proof of Theorem 4 to get that $\bigoplus_{k>0} \bigoplus_{\alpha_1, \dots, \alpha_k \in P\mathcal{U}} \mathcal{CH}_X(\alpha_1, \dots, \alpha_k)[k-1] \cong CH_X^\bullet(A)$). Since any factorization algebra is uniquely defined by its restriction to a factorizing basis, the second assertion follows easily from the first one applied to all $V \in \mathcal{B}$.

As a further corollary to Theorem 4, we can extend the Hochschild construction as a pullback and pushforward of factorization algebras in a particular setting. The pushforward construction of factorization algebras was described above Corollary 8. Following [8], there is also a **pullback** construction for factorization algebras given for an *open immersion* $f : N \rightarrow M$. For a factorization algebra \mathcal{F} on M , let $f^*\mathcal{F}$ be the factorization algebra on N given by $f^*\mathcal{F}(U) = \mathcal{F}(f(U))$ for all open subsets $U \subset N$ such that $f|_U : U \rightarrow f(U)$ is a homeomorphism, extended to a full factorization algebra.

Now, assume that X, Y, Z are topological spaces, and that there is an open immersion of $X \times Y \hookrightarrow Z$ of $X \times Y$ into Z . For a factorization algebra \mathcal{F} on Z , we define

the Hochschild factorization algebra with respect to X to be the factorization algebra $\mathbf{CH}_X(\mathcal{F})$ of \mathcal{F} on Y given by

$$\mathbf{CH}_X(\mathcal{F}) := (pr_Y)_* \circ f^*(\mathcal{F}), \quad \text{where } Y \xleftarrow{pr_Y} X \times Y \xrightarrow{f} Z.$$

Here $pr_Y : X \times Y \rightarrow Y$ denotes the projection to Y . This construction induces a functor, called $\mathbf{CH}_X : \text{Fac}_Z(\text{CDGA}_\infty) \rightarrow \text{Fac}_Y(\text{CDGA}_\infty)$, $\mathcal{F} \mapsto (pr_Y)_* \circ f^*(\mathcal{F})$ which satisfies the following naturality condition.

Corollary 11. *Assume that X , Y and Z all admit a basis of open sets which is a factorizing good cover. Under the functors $\mathcal{CH}_Y : \text{CDGA}_\infty \rightarrow \text{Fac}_Y(\text{CDGA}_\infty)$ and $\mathcal{CH}_Z : \text{CDGA}_\infty \rightarrow \text{Fac}_Z(\text{CDGA}_\infty)$ from Corollary 8, the functor \mathbf{CH}_X represents the functor CH_X^\bullet on CDGA_∞ , i.e., the following diagram commutes:*

$$\begin{array}{ccc} \text{CDGA}_\infty & \xrightarrow{CH_X^\bullet} & \text{CDGA}_\infty \\ \mathcal{CH}_Z \downarrow & & \downarrow \mathcal{CH}_Y \\ \text{Fac}_Z(\text{CDGA}_\infty) & \xrightarrow{\mathbf{CH}_X} & \text{Fac}_Y(\text{CDGA}_\infty) \end{array}$$

Proof. Let $A \in \text{CDGA}_\infty$ and \mathcal{W} be a basis and factorizing good cover by open subsets $U \times V \subset X \times Y$ such that U, V are contractible and $f|_{U \times V} : U \times V \rightarrow f(U \times V)$ is a homeomorphism. In particular, for $U \times V \in \mathcal{W}$, we have natural equivalences

$$f^*(\mathcal{CH}_Z(A))(U \times V) \cong \mathcal{CH}_Z(A)(f(U \times V)) \cong CH_{f(U \times V)}^\bullet(A) \cong A$$

since $f(U \times V)$ is contractible. Hence Corollary 10 implies that $f^*(\mathcal{CH}_Z(A))$ is strongly constant of type A and further that, for any contractible open $U \subset Y$,

$$\mathbf{CH}_X(\mathcal{CH}_Z(A))(U) \cong f^*(\mathcal{CH}_Z(A))(X \times U) \cong CH_{X \times U}^\bullet(A) \cong CH_X^\bullet(A).$$

Thus $\mathcal{CH}_X(\mathcal{CH}_Z(A))$ is a strongly constant factorization algebra on Y of type $CH_X^\bullet(A)$, hence, by Corollary 10.(2), is naturally equivalent to $\mathcal{CH}_Y(CH_X^\bullet(A))$.

4.3. Locally constant factorization algebras and E_n -algebras. If \mathcal{A} is a locally constant factorization algebra on $M \times D^{n-m}$, pushforward along the natural projection $\pi_1 : M \times D^{m-n} \rightarrow M$ induces a factorization algebra $\pi_{1*}(\mathcal{A})$ on M , which is locally constant. Given a monoidal $(\infty, 1)$ -category \mathcal{C} and a manifold X , we denote by $\text{Fac}_X^{lc}(\mathcal{C})$ **the (sub)category of locally constant factorization algebra** on X taking value in \mathcal{C} .

We recall the following Proposition which is essentially due to Lurie [24] and Costello [9].

Proposition 10. *The pushforward along $p : \mathbb{R}^n \rightarrow pt$ induces an equivalence of $(\infty, 1)$ -categories*

$$p_* : \text{Fac}_{\mathbb{R}^n}^{lc}(k\text{-Mod}_\infty) \xrightarrow{\simeq} E_n\text{-Alg}_\infty.$$

Proof. We sketch the proof. Details will appear elsewhere. Restricting to open sets homeomorphic to an euclidean disk, we obtain a tautological functor $for : Fac_{\mathbb{R}^n}^{lc}(k\text{-Mod}_{\infty}) \rightarrow N(\text{Disk}^{lc}(\mathbb{R}^n))\text{-Alg}$ where $N(\text{Disk}^{lc}(\mathbb{R}^n))\text{-Alg}$ stands for the full subcategory spanned by the locally constant $N(\text{Disk}(\mathbb{R}^n))$ -algebras in the sense of [24, Definition 5.2.4.7]. By [24, Theorem 5.2.4.9 and Example 5.2.4.3], the latter category is equivalent to $E_n\text{-Alg}_{\infty}$ and, under this equivalence, $for(\mathcal{F})$ identifies with the global section $\mathcal{F}(\mathbb{R}^n) \cong p_*(\mathcal{F})$ for any locally constant factorization algebra \mathcal{F} .

We define an inverse to p_* as follows. Let \mathcal{CVX} be the set of open convex subsets of \mathbb{R}^n , which is a factorizing basis. To an E_n -algebra E , one associates a \mathcal{CVX} -factorization algebra \mathcal{E} defined by $C \mapsto \mathcal{E}(C) := E$. As shown in [8], the \mathcal{CVX} -factorization algebra \mathcal{E} has an *unique* extension to a factorization algebra on \mathbb{R}^n iff it satisfies the Čech condition $\check{C}(\mathcal{U}, \mathcal{E}) \rightarrow \mathcal{E}(U)$ for any factorizing subcover $\mathcal{U} \subset \mathcal{CVX}$ of a convex open U . This follows again from [24, Section 5]. Indeed, by Theorem 5.3.4.10 in [24], the E_n -algebra E gives rise to a factorizable cosheaf \underline{E} on the Ran space $\text{Ran}(\mathbb{R}^n)$. It is easy to check that the factorizing cover \mathcal{U} gives rise to a cover of $\text{Ran}(U)$ precisely given by $P\mathcal{U}$. Since every open set in \mathcal{U} is convex, and $\int_C E \cong E$ for any convex open subset $C \subset U$, by [24, Theorem 5.3.4.14], we get a canonical equivalence $\check{C}(P\mathcal{U}, \underline{E}) \xrightarrow{\cong} \check{C}(\mathcal{U}, \mathcal{E})$ which makes the diagram

$$\begin{array}{ccccc} \underline{E}(\text{Ran}(U)) & \xleftarrow{\cong} & \check{C}(P\mathcal{U}, \underline{E}) & \xrightarrow{\cong} & \check{C}(\mathcal{U}, \mathcal{E}) \\ & \searrow \cong & & \swarrow & \\ & & E & & \end{array}$$

commutative (the top left equivalence follows from the fact that \underline{E} is a cosheaf on $\text{Ran}(U)$). Thus, $\check{C}(\mathcal{U}, \mathcal{E}) \rightarrow \mathcal{E}(U)$ is an equivalence and \mathcal{E} a \mathcal{CVX} -factorization algebra. We denote $q(E) := \mathcal{E}$ the induced factorization algebra over \mathbb{R}^n .

We need to check that $q(E)$ is locally constant. It is sufficient to prove that for any open D homeomorphic to an euclidean disk, the map $\mathcal{E}(D) \rightarrow \mathcal{E}(\mathbb{R}^n) \cong E$ is an equivalence. This follows from the Kister–Mazur Theorem [24, Theorem 5.2.1.5] which yields an isotopy between the inclusion $D \hookrightarrow \mathbb{R}^n$ and an homeomorphism of \mathbb{R}^n .

By construction, $p_* \circ q(E) = p_*(\mathcal{E}) \cong E$. Conversely, \mathcal{F} being locally constant, for any convex open set C , we have a canonical equivalence $\mathcal{F}(C) \cong \mathcal{F}(\mathbb{R}^n)$. It follows that the \mathcal{CVX} -factorization algebra defined by $p_*(\mathcal{F})$ is canonically equivalent to the restriction of \mathcal{F} to convex open sets. By uniqueness of the extension of factorization algebra defined on a factorization basis, $q \circ p_* \cong \text{id}$.

Lemma 4. *Let M be a manifold and $\pi_1 : M \times \mathbb{R}^d \rightarrow M$ the canonical projection. The pushforward by π_1 induces an equivalence of $(\infty, 1)$ -categories*

$$\pi_{1*} : Fac_{M \times \mathbb{R}^d}^{lc}(k\text{-Mod}_{\infty}) \xrightarrow{\cong} Fac_M^{lc}(E_d\text{-Alg}_{\infty})$$

In particular, if $\mathcal{F} \in Fac_{M \times \mathbb{R}^d}^{lc}(k\text{-Mod}_{\infty})$, then its factorization homology

$$HF(\mathcal{F}, M \times \mathbb{R}^d) = p_*(\mathcal{F})(pt) \cong p_* \circ \pi_{1*}(\mathcal{F})(pt) \cong HF(\pi_{1*}(\mathcal{F}), M)$$

is an E_d -algebra (here $p : X \rightarrow pt$ is the unique map).

Proof. Let $\pi_2 : M \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ be the second canonical projection. For any open set U in M , the restriction $\mathcal{A}|_{U \times \mathbb{R}^d}$ is a (locally constant) factorization algebra and $\pi_{2*}(\mathcal{A}|_{U \times \mathbb{R}^d})$ is a factorization algebra on \mathbb{R}^d . Note that $\pi_{2*}(\mathcal{A}|_{U \times \mathbb{R}^d})$ is locally constant. Indeed, let B be any (open, homeomorphic to a) ball $B \subset \mathbb{R}^n$ and denote $\text{Ball}^g(U)$ the set of geodesic convex open sets in U (for a choice of a metric on U). Then the family $\text{Ball}^g(U) \times B$ is a factorizing cover of $U \times B$. Since \mathcal{A} is locally constant, for any inclusion $B \hookrightarrow \tilde{B}$ of open sets (homeomorphic to) balls and geodesic convex open set O in U , the structure map $\mathcal{A}(O \times B) \rightarrow \mathcal{A}(O \times \tilde{B})$ is a quasi-isomorphism. Using that \mathcal{A} is a factorization algebra we get

$$\begin{aligned} \pi_{2*}(\mathcal{A}|_{U \times \mathbb{R}^d})(B) &\cong \mathcal{A}(U \times B) \\ &\cong \check{C}(\text{Ball}^g(U) \times B, \mathcal{A}) \\ &\cong \bigoplus_{k>0} \bigoplus_{\alpha_1 \dots \alpha_k \in P\text{Ball}^g(U)} \bigotimes_{i_1 \in \alpha_1, \dots, i_k \in \alpha_k} \mathcal{A}((U_{i_1} \cap \dots \cap U_{i_k}) \times B)[k-1] \\ &\cong \bigoplus_{k>0} \bigoplus_{\alpha_1 \dots \alpha_k \in P\text{Ball}^g(U)} \bigotimes_{i_1 \in \alpha_1, \dots, i_k \in \alpha_k} \mathcal{A}((U_{i_1} \cap \dots \cap U_{i_k}) \times \tilde{B})[k-1] \\ &\cong \check{C}(\text{Ball}^g(U) \times \tilde{B}, \mathcal{A}) \cong \pi_{2*}(\mathcal{A}|_{U \times \mathbb{R}^d})(\tilde{B}), \end{aligned}$$

which proves that $\pi_{2*}(\mathcal{A}|_{U \times \mathbb{R}^d})$ is locally constant (since the above composition is the structure map $\pi_{2*}(\mathcal{A}|_{U \times \mathbb{R}^d})(B) \rightarrow \pi_{2*}(\mathcal{A}|_{U \times \mathbb{R}^d})(\tilde{B})$). Hence, by Proposition 10, $\pi_{2*}(\mathcal{A}|_{U \times \mathbb{R}^d})$ is equivalent to an E_d -algebra A_U and there are canonical equivalences

$$A_U \cong \pi_{2*}(\mathcal{A}|_{U \times \mathbb{R}^d})(\mathbb{R}^d) \cong \mathcal{A}(U \times \mathbb{R}^d).$$

Since $\pi_{1*}(\mathcal{A})(U) = \mathcal{A}(U \times \mathbb{R}^d) \cong A_U$, it follows that for each open subset $U \in \text{Op}(M)$, $\pi_{1*}(\mathcal{A})(U)$ is an E_d -algebra. Since \mathcal{A} is a (pre)factorization algebra on $M \times \mathbb{R}^d$, the structure maps

$$\begin{aligned} \mu_{U_1, \dots, U_k, V} : \pi_{1*}(\mathcal{A})(U_1) \otimes \dots \otimes \pi_{1*}(\mathcal{A})(U_k) &\cong \mathcal{A}(U_1 \times \mathbb{R}^d) \otimes \dots \otimes \mathcal{A}(U_k \times \mathbb{R}^d) \\ &\longrightarrow \mathcal{A}(V \times \mathbb{R}^d) \cong \pi_{1*}(\mathcal{A})(V) \end{aligned}$$

are maps of E_d -algebras. Further, since \mathcal{A} is a factorization algebra and locally constant, it follows that $\pi_{1*}(\mathcal{A})(U)$ belongs to $\text{Fac}_M^{lc}(E_d\text{-Alg}_\infty)$.

Now we build an inverse of π_{1*} . Consider \mathcal{B} in $\text{Fac}_M^{lc}(E_d\text{-Alg}_\infty)$. For any $U \in \text{Op}(M)$, $\mathcal{B}(U)$ is an E_d -algebra (compatible with the prefactorization algebra structure map) and thus defines canonically a locally constant factorization algebra on \mathbb{R}^d : $\text{Op}(\mathbb{R}^d) \ni V \mapsto \mathcal{B}(U)(V)$. A basis of neighborhood of $M \times \mathbb{R}^d$ is given by the products $U \times V \in \text{Op}(M) \times \text{Op}(\mathbb{R}^d)$. In order to extend \mathcal{B} to a factorization algebra on $M \times \mathbb{R}^d$, it is enough (by [8]) to prove that the rule $(U \times V) \mapsto \mathcal{B}(U)(V)$ defines an $\text{Op}(M) \times \text{Op}(\mathbb{R}^d)$ -factorization algebra where $\text{Op}(M) \times \text{Op}(\mathbb{R}^d)$ is the cover of $M \times \mathbb{R}^d$ obtained by taking the products of open sets. The latter follows from the fact that the E_d -algebra structure is natural with respects to the inclusions $\mu_{U_1, \dots, U_n, \tilde{U}}$ of pairwise disjoint open subsets of $\tilde{U} \in \text{Op}(M)$. Hence the rule $(U \times V) \mapsto \mathcal{B}(U)(V)$ extends to give a factorization algebra $E(\mathcal{B})$ on $M \times \mathbb{R}^d$. It is immediate by construction that $E(\mathcal{B})$ is locally constant and functorial in \mathcal{B} .

It remains to prove that $E : \text{Fac}_M^{lc}(E_d\text{-Alg}_\infty) \rightarrow \text{Fac}_{M \times \mathbb{R}^d}^{lc}(k\text{-Mod}_\infty)$ is a natural inverse to π_{1*} . Recall from above that the E_d -algebra structure on $\pi_{1*}(\mathcal{A})(U)$ is

the one of the E_d -algebra $\mathcal{A}(U \times \mathbb{R}^d)$, which corresponds to the factorization algebra $V \mapsto \pi_{2*}(\mathcal{A}|_{U \times \mathbb{R}^d}(V)) \cong \mathcal{A}(U \times V)$. It follows that there is a natural equivalence $E(\pi_{1*}(\mathcal{A}))(U \times V) \cong \mathcal{A}(U \times V)$ in $k\text{-Mod}_\infty$. Further there are natural equivalences (in $E_d\text{-Alg}_\infty$) $\pi_{1*}(E(\mathcal{B}))(U) \cong E(\mathcal{B})(U \times \mathbb{R}^d) \cong \mathcal{B}(U)$. The lemma now follows.

Let \mathcal{A} be a locally constant factorization algebra on a manifold M and assume that there is a codimension 1 submanifold (possibly with corners) N of M with a trivialization $N \times D^1$ of its neighborhood such that M is decomposable as $M = X \cup_{N \times I} Y$ where X, Y are submanifolds (with corners) of M glued along $N \times D^1$. According to Lemma 4 above $HF(\mathcal{A}|_{N \times D^1})$ is an E_1 -algebra.

Lemma 5 (Excision for locally constant factorization algebras). *$HF(\mathcal{A}|_X)$ and $HF(\mathcal{A}|_Y)$ are right and left $HF(\mathcal{A}|_{N \times D^1})$ -modules and further,*

$$HF(\mathcal{A}) \cong HF(\mathcal{A}|_X) \underset{HF(\mathcal{A}|_{N \times D^1})}{\overset{\mathbb{L}}{\otimes}} HF(\mathcal{A}|_Y).$$

Proof. Since \mathcal{A} is locally constant, the canonical map $\mathcal{A}(X \setminus (N \times [t, 1])) \rightarrow \mathcal{A}(X)$ is an equivalence for all $t \in D^1$. This follows, since for any open set of the form $U \times (a, b) \subset N \times D^1$, where U is homeomorphic to a ball, there is a natural equivalence $\mathcal{A}(U \times (a, b)) \cong \mathcal{A}(U \times (a', b'))$ for any $a' < a < b < b'$ and that the open sets of the form $U \times (a, b)$ form a factorizing cover of $N \times D^1$. Similarly, we have natural equivalences of E_1 -algebras $\mathcal{A}(N \times (a, b)) \xrightarrow{\cong} N \times D^1$ (as in the proof of Lemma 4).

Let U be an open set in X and V_1, \dots, V_k be (not necessarily disjoint) open subsets in N . Then for any sequence of pairwise disjoint open intervals I_0, I_1, \dots, I_k in D^1 (where we assume $I_0 = (-1, t_0)$ for some $t_0 \in D^1$), the open $V_i \times I_i$ ($i = 1 \dots k$) are pairwise disjoint and disjoint from $X \setminus (N \times [t_0, 1])$. To shorten notation, we denote $X_{t_0} := X \setminus (N \times [t_0, 1])$. The structure maps of a prefactorization algebras yield a map

$$\begin{array}{ccc} \mathcal{A}(X) \otimes \mathcal{A}(N \times D^1)^{\otimes n} & \xrightarrow{\cong} & \mathcal{A}(X_{t_0}) \otimes \mathcal{A}(N \times I_1) \otimes \dots \otimes \mathcal{A}(N \times I_k) \\ & \searrow & \downarrow \mu_{X_{t_0}, N \times I_1 \dots N \times I_k, X} \\ & & \mathcal{A}(X) \end{array}$$

This map is natural with respect to the prefactorization algebra structure of \mathcal{A} and $\mathcal{A}_{N \times D^1}$, hence induces a structure of right $HF(\mathcal{A}_{N \times D^1}) \cong \mathcal{A}(N \times D^1)$ -module on $HF(\mathcal{A}|_X) \cong \mathcal{A}(X)$. A similar argument applies to $HF(\mathcal{A}|_Y)$.

The open sets $X_{t_0}, Y_s := Y \setminus (N \times (-1, s])$ and $N \times (a, b)$ (where $t_0, s, a < b \in D^1$) forms a factorizing cover \mathcal{N} of M and we also denote $\tilde{\mathcal{N}}$ the induced factorizing cover of $N \times D^1$. A finite sequence of pairwise disjoint open sets in \mathcal{N} is just a sequence $X_{t_0}, N \times (t_1, t_2), \dots, N \times (t_m, t_{m+1}), Y_{t_m}$ for $-1 < t_0 < \dots < t_{m+2} < 1$. Note that the complexes $\mathcal{A}(N \times (t_i, t_{i+1}))$ are canonically equivalent to $\pi_{2*}(\mathcal{A})((t_i, t_{i+1}))$, where $\pi_2 : N \times D^1 \rightarrow D^1$ is the projection on the second factor. Since $\mathcal{A}(X_{t_0}) \cong \mathcal{A}(X)$, $\mathcal{A}(Y_{t_{m+2}}) \cong \mathcal{A}(Y)$, we have

$$HF(\mathcal{A}) \cong \check{C}(\mathcal{N}, \mathcal{A}) \cong \mathcal{A}(X) \otimes \check{C}(\tilde{\mathcal{N}}, \mathcal{A}|_{N \times D^1}) \otimes \mathcal{A}(Y).$$

Note that the canonical map $p : M \rightarrow pt$ factors as $M \xrightarrow{q} [-1, 1] \rightarrow pt$ where q is the map identifying $X \setminus (N \times D^1)$ with $\{-1\}$, $Y \setminus (N \times D^1)$ with $\{1\}$ and projecting $N \times D^1$ onto $D^1 = (-1, 1)$ by the second projection. Then the factorization homology $HF(\mathcal{A}) \cong p_*(\mathcal{A}) \cong p_*(q_*(\mathcal{A}))$ identifies with the factorization homology of the locally constant factorization algebra $q_*(\mathcal{A})$ on the closed interval $[-1, 1]$ and further $\check{C}(\mathcal{N}, \mathcal{A}) \cong \check{C}(\mathcal{I}, q_*(\mathcal{A}))$ where \mathcal{I} is the (factorizing) cover of $[-1, 1]$ given by the intervals. Thus we are left to the case of a (locally constant) factorization algebra on $[-1, 1]$ which assign the E_1 -algebra $\mathcal{A}(N \times D^1)$ to any open interval (a, b) , and the modules $\mathcal{A}(X)$ to $[-1, t)$ and $\mathcal{A}(Y)$ to $(s, 1]$ (with respect to the modules structures defined in the beginning of the proof). It is stated in [8, 9] that its factorization homology is the (derived) tensor product $\mathcal{A}(X) \otimes_{\mathcal{A}(N \times D^1)}^{\mathbb{L}} \mathcal{A}(Y)$. Indeed, one can prove that the Čech complex $\check{C}(\mathcal{I}, q_*(\mathcal{A}))$ is equivalent to the two-sided Bar construction $Bar(\mathcal{A}(X), \mathcal{A}(N \times D^1), \mathcal{A}(Y))$ as we now explain.

First, by strictification we can replace the E_1 -algebra and modules by differential graded associative ones so that we are left to the case of a factorization algebra \mathcal{F} on $[-1, 1]$ which, to any open interval $(u, s) \subset (-1, 1)$ associates $\mathcal{F}((u, s)) = A$, to $[-1, t) \subset [-1, 1)$ associates $\mathcal{F}([-1, t)) = M$ and to $(s, 1] \subset (-1, 1]$ associates $\mathcal{F}((s, 1]) = N$, where A is a differential graded algebra, and N, M respectively left and right dg- A -modules. We wish to find a factorizing cover \mathcal{U} of $[-1, 1]$ such that $\check{C}(\mathcal{U}, \mathcal{F})$ is quasi-isomorphic to the two sided Bar construction $Bar(M, A, N)$. Since $Bar(M, A, N) \cong M \otimes_A Bar(A, A, A) \otimes_A N$, it is enough to prove the result when N, M are A endowed with its canonical dg-modules structures, which we now assume. Let \mathcal{U} be the factorizing cover given by all opens $U_t := [-1, 1] \setminus \{t\}$ where $t \in [-1, 1]$ (in other words by the complement of a singleton). Any two such opens intersects non-trivially so that the set PU are singletons. We have $\mathcal{F}(U_t) \cong \mathcal{F}([-1, t)) \otimes \mathcal{F}((t, 1])$ which is $A \otimes A$ if $t \neq \pm 1$ and is $A \otimes k$ or $k \otimes A$ if $t = 1$ or $t = -1$; more generally, $\mathcal{F}(U_{t_0}, \dots, U_{t_n}, U_{\pm 1}) \cong \mathcal{F}(U_{t_0}, \dots, U_{t_n}) \otimes k$. Further, if $-1 < t_0 < \dots < t_n < 1$, then $\mathcal{F}(U_{t_0}, \dots, U_{t_n}) \cong A \otimes A^{\otimes n} \otimes A$ and the structure map $\mathcal{F}(U_{t_0}, \dots, U_{t_n}) \rightarrow \mathcal{F}(U_{t_0}, \dots, \widehat{U_{t_i}}, \dots, U_{t_n})$ is given by the multiplication $a_0 \otimes \dots \otimes a_{n+1} \mapsto a_0 \otimes \dots \otimes (a_i a_{i+1}) \otimes \dots \otimes a_{n+1}$. Hence the Čech complex $\check{C}(\mathcal{U}, \mathcal{F})$ is a (parametrized by t) version of the two sided Bar construction with coefficients in A and can be shown to be quasi-isomorphic to A similarly. To see this, note that the multiplication in A induces canonical maps

$$\mathcal{F}(U_t) \cong \mathcal{F}([-1, t)) \otimes \mathcal{F}((t, 1]) \rightarrow \mathcal{F}([-1, 1)) \otimes \mathcal{F}((-1, 1]) \cong A \otimes A \rightarrow A$$

such that the induced composition $\bigoplus_{U_r, U_s \in PU} \mathcal{F}(U_r, U_s)[1] \rightarrow \bigoplus_{U_t \in PU} \mathcal{F}(U_t)[0] \rightarrow A$ is

null. Hence, we get a canonical map of chain complexes $\eta : \check{C}(\mathcal{U}, \mathcal{F}) \rightarrow A$ which we claim is a quasi-isomorphism. Indeed, the map $\kappa : A \cong \mathcal{F}(U_1) \hookrightarrow \bigoplus_{U_t \in PU} \mathcal{F}(U_t)[0] \hookrightarrow$

$\check{C}(\mathcal{U}, \mathcal{F})$ is a retract of η , i.e., $\eta \circ \kappa = id_A$. There is an homotopy operator h on $\check{C}(\mathcal{U}, \mathcal{F})$ defined, on $\mathcal{F}(U_{t_0}, \dots, U_{t_n})[n]$, by

$$\sum_{i=0}^n (-1)^i s_i^{t_0, \dots, t_n} : \mathcal{F}(U_{t_0}, \dots, U_{t_n})[n] \longrightarrow \bigoplus_{U_{r_0}, \dots, U_{r_{n+1}} \in PU} \mathcal{F}(U_{r_0}, \dots, U_{r_{n+1}})[n+1]$$

where, for $0 \leq i \leq n-1$, $s_i^{t_0, \dots, t_n}$ is the (suspension of the) identity map $\mathcal{F}(U_{t_0}, \dots, U_{t_n})[n] \rightarrow \mathcal{F}(U_{t_0}, \dots, U_{t_n})[n+1] \cong \mathcal{F}(U_{t_0}, \dots, U_{t_i}, U_{t_i}, \dots, U_{t_n})[n+1]$ followed by the inclusion in the Čech complex while the map $s_n^{t_0, \dots, t_n}$ is the (suspension of the) identity map $\mathcal{F}(U_{t_0}, \dots, U_{t_n})[n] \rightarrow \mathcal{F}(U_{t_0}, \dots, U_{t_n})[n+1] \cong \mathcal{F}(U_{t_0}, \dots, U_{t_n}, U_1)[n+1]$ (followed by the inclusion in the Čech complex). It is straightforward that $dh + hd = id - \kappa \circ \eta$ (where d is the differential on $\check{C}(\mathcal{U}, \mathcal{F})$) which proves that $\eta : \check{C}(\mathcal{U}, \mathcal{F}) \rightarrow A$ is a quasi-isomorphism hence, that $\mathcal{F}([-1, 1])$ is the two-sided bar construction $A \cong \text{Bar}(A, A, A)$ which concludes the proof.

5. Relationship with Topological Chiral Homology

5.1. Review of topological chiral homology à la Lurie. Let A be an E_n -algebra and M a manifold of dimension m which is (stably) n -framed, that is a manifold of dimension m equipped with a framing of $M \times D^{n-m}$. The **topological chiral homology** of M with coefficients in A was defined in [11, 24, 25] and will be denoted $\int_M A$. Note that this concept *does* depend on the framing in general, even though it is not explicit in the notation⁵. Further $\int_M A$ is an E_{n-m} -algebra in general, see [24, 25]. We refer to the above references as well as to [1, 2, 11] for a precise definition. If X is a manifold, let $N(\text{Disj}(X))$ be the ∞ -category associated to the poset given by finite disjoint union of open sets in X homeomorphic to an euclidean disk, ordered by inclusion. According to Lurie [24, Remark 5.3.2.7] we have, roughly, that

Definition 4. Let M be an n -framed manifold of dimension m and A an E_n -algebra. The topological chiral homology $\int_M A$ is the colimit $\varinjlim \psi_M$ with $\psi_M : N(\text{Disj}(M \times D^{n-m})) \rightarrow k\text{-Mod}_\infty$ the diagram given by the formula

$$\psi_M(V_1 \cup \dots \cup V_n) = \int_{V_1} A \otimes \dots \otimes \int_{V_n} A \cong A \otimes \dots \otimes A \quad (13)$$

where V_1, \dots, V_n are disjoint open sets homeomorphic to a ball (the latter equivalence follows from [24, Example 5.3.2.8]).

For our purpose, among the properties satisfied by $\int_M A$, we will mainly need the gluing property given in Proposition 11 below and the fact that $\int_{pt} A \cong A$. [25, Theorem 4.1.24].

Remark 12. The models for E_n -algebras that we are considering are given by E_n -(∞ -) operads as introduced in [24, Section 5.1] in the symmetric monoidal $((\infty, 1)\text{-category})$ $(k\text{-Mod}_\infty, \otimes)$. The category of E_n -algebras is symmetric monoidal ([24, Section 5.1.5], [23, Section 1.8]) and, furthermore, there is a commutative diagram of operads

$$\begin{array}{ccccccc} E_1 & \hookrightarrow & E_2 & \hookrightarrow & \dots & \hookrightarrow & E_n & \hookrightarrow & \dots \\ & & & \searrow j_2 & & & \swarrow j_n & & \\ & & & & & & Com & & \end{array} \quad (14)$$

j_1 (from E_1 to Com)

where Com is the operad of commutative (differential graded) algebras such that all maps are monoids. Note that most models for E_n -algebras come with such monoidal

⁵ note that we also do not write the factor D^{n-m} in the notation.

properties and also as nested sequences (for instance, this is the case for the models based on the Barratt–Eccles operad [4]). In particular, a commutative differential graded algebra A is naturally an E_d -algebra for any integer d . We write $j_d^*(A)$ for the E_d -algebra structure on A induced by the map of operads $j_d : E_d \rightarrow \text{Com}$ whenever we want to put emphasis on this E_d -algebra structure.

Likewise, any E_n -algebra A is naturally an E_d algebra for $d \leq n$. Furthermore, we say that an A -module M has a *compatible* structure of E_d -algebras ($d \leq n$) if the structure maps of the module structure are maps of E_d -algebras, where A is equipped with its natural E_d -algebra induced by the diagram of operads (14).

Remark 13. With the exception of Sect. 5.3.2, this paper only deals with a fixed E_n -algebra A , which, when M is framed, is an example of a locally constant $N(\text{Disk}(M))$ -algebra in the sense of Lurie [24], and for which topological chiral homology can be defined, too. In particular, we will show that A also defines canonically a locally constant factorization algebra on M in the sense of Costello and Gwilliam [8, 9]. This also means, that $\int_M A$ computes the global sections of a natural cosheaf defined on the Ran space of M (see [24, Section 5.3.2]).

One of the main consequences of the interpretation [25] of topological chiral homology as an invariant produced by an (extended) topological field theory in some appropriate monoidal (∞, n) -category is the following excision property.

Proposition 11 (Gluing for topological chiral homology). *Let M be an n -framed manifold (possibly with corners) of dimension m , (i.e. $M \times D^{n-m}$ is framed). Assume that there is a codimension 1 submanifold (possibly with corners) of M of the form $N \times I^{m-1-j}$ (for some $0 \leq j \leq m-1$) with a trivialization $N \times I^{m-j}$ of its neighborhood and that M is decomposable as $M = X \cup_{N \times I^{m-j}} Y$ where X, Y are submanifolds (with corners) of M glued along $N \times I^{m-j}$. We endow X, Y and N with the n -framing induced from M . Let A be an E_n -algebra. Then*

- $\int_N A$ is an E_{n-j} -algebra.
- $\int_M A$, $\int_X A$, and $\int_Y A$ are E_{n-m} -algebras. Further $\int_X A$ and $\int_Y A$ are also modules over the E_{n-j} -algebra $\int_N A$.
- The above module and algebra structures are compatible. Note that this uses the once and for all fixed telescopic sequence (14) of models for E_n -operads.
- There is a natural equivalence of E_{n-m} -algebras

$$\int_X A \otimes_{\int_N A} \int_Y A \xrightarrow{\sim} \int_M A.$$

Proof. This is explained in [24, Section 5.3.4] and [25, Section 4.1], also see the proof of [11]. It is also an immediate consequence of Lemma 8 below, in the case where the manifolds are framed since, for a n -framed manifold X , an E_n -algebra yields canonically an $\mathbb{E}[X]$ -algebra.

Lemma 6. *Let A be an E_n -algebra and $(M_i)_{i \in I}$ a family of n -framed manifolds of dimension m . There is a natural equivalence of E_{n-m} -algebras*

$$\lim_{\substack{F \subset J \\ F \text{ finite}}} \left(\bigotimes_{f \in F} \int_{M_f} A \right) \xrightarrow{\sim} \int_{\coprod_{i \in I} M_i} A.$$

Proof. We set $M = \coprod_{i \in I} M_i$ and, for any finite subset F of I , we denote $M_F := \coprod_{f \in F} M_f$. The inclusion $M_F \subset M$ yields a canonical map $N(\text{Disj}(M_F \times D^{n-m})) \rightarrow N(\text{Disj}(M \times D^{n-m}))$. Since an object in $N(\text{Disj}(M \times D^{n-m}))$ is a *finite* disjoint union of connected open sets in M , every object in $N(\text{Disj}(M \times D^{n-m}))$ lies in some $N(\text{Disj}(M_F \times D^{n-m}))$ for a finite F . Hence we have an equivalence $N(\text{Disj}(M \times D^{n-m})) \cong \varinjlim_{F \text{ finite}} N(\text{Disj}(M_F \times D^{n-m}))$ of ∞ -categories and, by Definition 4, a natural equivalence

$$\int_M A \cong \varinjlim \psi_M \cong \varinjlim_{F \text{ finite}} \psi_{M_F} \cong \varinjlim_{F \text{ finite}} \int_{M_F} A.$$

Now the lemma follows from [24, Theorem 5.3.3.1] which gives a natural equivalence $(\bigotimes_{f \in F} \int_{M_f} A) \xrightarrow{\sim} \int_{M_F} A$ for all finite F .

5.2. Locality axiom and the equivalence of topological chiral homology with higher Hochschild functor for CDGAs. In view of Proposition 11 and Theorem 1, Morse theory (or any triangulation) suggests the following result, which is the main result of this section.

Theorem 5. *Let M be a manifold of dimension m endowed with a framing of $M \times D^k$ and A be a differential graded commutative algebra viewed as an E_{m+k} -algebra. Then, the topological chiral homology of M with coefficients in A , denoted by $\int_M A$ is equivalent to $CH_M^\bullet(A)$ viewed as an E_k -algebra (in other words to $j_k^*(CH_M^\bullet(A))$).*

In particular, topological chiral homology $\int_M A$ with coefficient in a CDGA A is always equivalent to a CDGA and is defined for any manifold M .

This theorem is similar to [24, Theorem 5.3.3.8] (with the difference that we assume M to be smooth and keep track of the E_k -algebra structure). In this section, we wish to prove it by using a straightforward geometrical approach based on the gluing property. Indeed, the key idea is to use handle decomposition which is very appropriate to deal with manifolds and the definition of topological chiral homology. However, note that, with respect to Hochschild chains, a representation of M as a CW complex is already nice enough.

Before proving Theorem 5, we recall a few facts on the handle decompositions. Let M be a smooth manifold of dimension m . A *handle decomposition* of M is a sequence $\emptyset \subset M_0 \subset \dots \subset M_m = M$, where each M_j is obtained by attaching j -handles to M_{j-1} , see [27]. That is gluing a copy of $H^j = D^j \times D^{m-j}$ using the attaching map $S^{j-1} \times D^{m-j} \rightarrow \partial M_{j-1}$ which is assumed to be an embedding. In particular all handles of same dimension are attached using diffeomorphisms.

We can achieve such a handle decomposition for M using Morse theory as follows. Let $f : M \rightarrow \mathbb{R}$ be a Morse function with critical points p_1, \dots, p_k numbered in a way that $f(p_1) < \dots < f(p_k)$. Choose a_0, \dots, a_k such that $a_0 < f(p_1) < a_1 < \dots < a_{k-1} < f(p_k) < a_k$. Now it is sufficient to note that $f^{-1}([a_{i-1}, a_i])$ is diffeomorphic to attaching a j -handle to $f(a_{i-1}) \times [0, 1]$, where j is the index of the critical point p_i , i.e. the number of the negative eigenvalues of the Hessian of f at that critical point. For example a torus with the height function is first given by attaching a $D^0 \times D^2$ to the empty set, then attaching a $D^1 \times D^1$ (think of it as a thin ribbon) to the boundary of the

previous D^2 . Then attaching another ribbon to the boundary of the previous ribbon, and the finally attaching a $D^2 \times D^0$ to what has been obtained along the boundary.

Lemma 7. *Let M be an n -framed manifold and N be an n -framed manifold obtained from M by attaching a countable sequence of handles $(H_i)_{i \in \mathbb{N}}$. For any $n \in \mathbb{N}$, we write X_k for the result of attaching the first k -handles to M . For any E_n -algebra A , there is a natural equivalence*

$$\varinjlim_{k \in \mathbb{N}} \int_{X_k} A \xrightarrow{\sim} \int_N A.$$

Proof. We may assume $\dim(M) = n$. The maps $X_k \rightarrow X_{k+1}$ yield a diagram

$$N(\text{Disj}(M)) \rightarrow N(\text{Disj}(X_1)) \rightarrow \cdots \rightarrow N(\text{Disj}(X_k)) \rightarrow \cdots \rightarrow N(\text{Disj}(N)) \quad (15)$$

of faithful maps, hence canonical maps $\varinjlim_{k \in \mathbb{N}} N(\text{Disj}(X_k)) \rightarrow N(\text{Disj}(N))$ and

$$\varinjlim_{k \in \mathbb{N}} \int_{X_k} A \cong \varinjlim_{k \in \mathbb{N}} \left(\varinjlim \psi_{X_k} \right) \rightarrow \varinjlim \psi_N \cong \int_N A \quad (16)$$

(using the notation introduced in Definition 4). Note that $\varinjlim_{k \in \mathbb{N}} \left(\varinjlim \psi_{X_k} \right)$ can be identified with the colimit $\varinjlim \tilde{\psi}_N$ given informally by the diagram

$$\tilde{\psi}_N(V_1^k \cup \cdots \cup V_j^k) = \int_{V_1^k} A \otimes \cdots \otimes \int_{V_j^k} A$$

where the V_i^k are disjoint open subsets of X_k homeomorphic to a ball (here k is not fixed).

Since A is a fixed E_n -algebra, it is in particular an $\mathbb{E}(X)$ -algebra (in the sense of [24, Section 5.2.4]) for any n -framed manifold X . In particular, by [24, Theorem 5.2.4.9] (also see Proposition [24, Proposition 5.3.2.13]), if $U \subset V$ are two open subsets of X which are homeomorphic to a ball, then the induced map $\int_U A \rightarrow \int_V A$ is an equivalence.

Now, let V be an open subset of N which is homeomorphic to a ball. Since $N = \varinjlim_{k \in \mathbb{N}} X_k$, there exists a k such that $V \cap X_k$ is a non-empty open subset of X_k , and

thus contains an open ball V^k in X_k which is lying in V too. In particular the natural map $\int_{V^k} A \rightarrow \int_V A$ is an equivalence. It follows that, given a finite set V_1, \dots, V_n of open subsets homeomorphic to a ball in N , we can find an integer k big enough and open sets $V_1^k \subset V_1, \dots, V_n^k \subset V_n$ in X_k which are homeomorphic to a ball, yielding a natural equivalence

$$\tilde{\psi}_N(V_1^k, \dots, V_n^k) = \int_{V_1^k} A \otimes \cdots \otimes \int_{V_n^k} A \cong \int_{V_1} A \otimes \cdots \otimes \int_{V_n} A = \psi_N(V_1, \dots, V_n).$$

This proves the cofinality of the functor $N(\widehat{\text{Disj}}(N)) \hookrightarrow N(\text{Disj}(N))$ induced by the natural inclusion. Here we have denoted $\widehat{\text{Disj}}(N)$ the partially ordered set of open subsets of N which are homeomorphic to $F \times \mathbb{R}^n$ for a finite set F and included in some X_k (where k is not fixed). Passing to colimits, we get that the map $\varinjlim_{k \in \mathbb{N}} \left(\varinjlim \psi_{X_k} \right) \cong \varinjlim \tilde{\psi}_N \rightarrow$

$\varinjlim \psi_N$ is an equivalence and thus the canonical map (16) is an equivalence as well.

Proof (Proof of Theorem 5). Let us sketch the key idea of the proof first: by the value on a point axiom, both topological chiral homology and higher Hochschild chains agree on a point and further on any disk D^k (up to neglect of structure). Using handle decompositions, one can chop manifolds on disks which are glued along their boundaries. Since both topological chiral homology and higher Hochschild chains satisfy a similar gluing axiom (and also behave the same way under disjoint unions), one then can lift the natural equivalence for disks to any manifold using handle decompositions. We now make the above scheme precise.

Assume M is compact. Let us choose a generic Morse function on M and the associated handle decomposition $\emptyset \subset M_0 \subset \dots \subset M_m = M$ of M . Then $\emptyset \subset M_0 \times D^k \subset \dots \subset M_m \times D^k = M \times D^k$ is a handle decomposition of $M \times D^k$. That is $(M \times D^k)_j = M_j \times D^k$ where we replace each j -handle $H^j = D^j \times D^{m-j}$ attached to M_{j-1} by the j -handle $D^j \times D^{m+k-j} \cong D^j \times (D^{m-j} \times D^k)$ attached to $M_{j-1} \times D^k = (M \times D^k)_{j-1}$. The $(m+k)$ -framing of M induces an $(m+k)$ -framing of each $M_j \times D^k = (M \times D^k)_j$.

By homotopy invariance of Hochschild chains, one has an equivalence of CDGAs $CH_{M \times D^d}(A) \cong CH_M(A)$ (for any integer d). Further, from diagram (14) we deduce that, for any CDGA B , one has $j_k^*(B) \cong \iota_d^*(j_{k+d}^*(B))$ where $\iota_d : E_k \hookrightarrow E_{k+d}$ is the natural map. Since, for any E_{m+k} -algebra B , one has $\int_{D^m} B \cong B$ viewed as an E_k -algebra, the result of Theorem 5 holds for all disks.

We now prove by induction that it holds for all $((m+d)$ -framed) spheres S^m and E_{m+d} -algebra A . For $S^0 = pt \coprod pt$, it follows from Theorem 1 and [24, Theorem 5.3.3.1] that $\int_{S^0} A \cong A \otimes A \cong CH_{S^0}(A)$ (as E_d -algebras). Now, assume the result for S^{m-1} and $m \geq 1$. We have a decomposition of the m -sphere S^m as $S^m \cong D^m \cup_{S^{m-1}} D^m$ as in the assumption of Proposition 11. Since this decomposition is also a homotopy pushout, it follows from the induction hypothesis, Proposition 11 and Theorem 1.(3) that there are natural equivalences

$$\int_{S^m} A \cong A \underset{\int_{S^{m-1}} A}{\otimes} A \cong CH_{D^m}^\bullet(A) \underset{CH_{S^{m-1}}^\bullet(A)}{\otimes} CH_{D^m}^\bullet(A) \cong CH_{S^m}^\bullet(A)$$

of E_d -algebras which finishes the induction step.

Clearly, M_0 is a disjoint union $M_0 = \coprod_{I_0} D^m$ of finitely many m -dimensional balls (here I_0 is the set indexing the various disks in M_0). Using again that, for any E_{m+k} -algebra B , one has $\int_{D^m} B \cong B$ viewed as an E_k -algebra, we get a natural equivalence of E_k -algebras $\int_{M_0} A \cong \bigotimes_{I_0} j_k^*(A) \cong j_k^*(\bigotimes_{I_0} A)$ since $\int_M \coprod_N A \cong \int_M A \otimes \int_N A$ by [24, Theorem 5.3.3.1], the set I_0 is finite and j_k is monoidal (that is commutes with the diagonals of the E_k -operads). Further $CH_{D^m}^\bullet(A) \cong CH_{pt}^\bullet(A) \cong A$ and, by Theorem 1.(1) and (2), there is a natural equivalence $CH_{M_0}^\bullet(A) \cong \bigotimes_{I_0} A$ of CDGAs. Hence the theorem is proved for M_0 .

By assumption M_1 is obtained by attaching finitely many 1-handles $H_1^1, \dots, H_{i_1}^1$ (the sequence may be empty) to M_0 . Choosing appropriate tubular neighborhoods for the image of $\partial D^1 \times D^{m-1}$ in M_0 and gluing it with $(\partial D^1 \times D^{m-1}) \times [0, \varepsilon] \cong (\partial D^1 \times [0, \varepsilon] \times D^{m-1}) \subset H_1^1$, we can assume that the result $M_0 \cup_{\partial D^1 \times D^{m-1}} H_1^1$ of attaching H_1^1 to M_0 satisfies the assumption of Proposition 11. Then, by Proposition 11 we have a natural equivalence of E_k -algebras:

$$\int_{M_0} A \underset{\int_{\partial D^1 \times D^{m-1}} A}{\otimes} \int_{H_1^1} A \cong \int_{M_0 \cup_{\partial D^1 \times D^{m-1}} H_1^1} A. \quad (17)$$

By Theorem 1.(3), we also have a natural equivalence of CDGAs (and thus of the underlying E_k -algebras)

$$CH_{M_0}^\bullet(A) \cong_{CH_{\partial D^1 \times D^{m-1}}^\bullet(A)}^{\mathbb{L}} CH_{H_1}^\bullet(A) \cong CH_{M_0 \cup_{\partial D^1 \times D^{m-1}} H_1}^\bullet(A). \quad (18)$$

We have already seen that we have natural equivalences $CH_{M_0}^\bullet(A) \cong \int_{M_0} (A)$ of E_k -algebras and similarly for $\partial D^1 \times D^{m-1}$ and $D^1 \times D^{m-1}$ in place of M_0 (since those manifolds are disks). Combining these equivalences with those given by the identities (17) and (18), we get a natural equivalence

$$\int_{M_0 \cup_{\partial D^1 \times D^{m-1}} H_1} A \cong CH_{M_0 \cup_{\partial D^1 \times D^{m-1}} H_1}^\bullet(A).$$

Attaching more 1-handles, we inductively get a natural equivalence $\int_{M_1} A \cong CH_{M_1}^\bullet(A)$ of E_k -algebras.

We proceed the same for attaching $j \geq 2$ -handles (by induction on j). The proof is identical to the 1-handles case once we notice that there are also a natural equivalence $\int_{\partial D^j \times D^{m-j}} A \cong CH_{\partial D^j \times D^{m-j}}^\bullet(A)$. The later follows from the natural equivalences relating topological chiral homology and higher Hochschild chains of spheres (with value in A) proved above. Indeed, $\partial D^j \times D^{m-j} \cong S^{j-1} \times D^{m-j}$ and there is a natural equivalence $\int_{S^{j-1} \times D^{m-j}} A \cong i_{m-j}^* (\int_{S^{j-1}} A)$ where $i_{m-j} : E_{k+1} \hookrightarrow E_{m+k-j+1}$ is the canonical map (neglecting part of the structure). Similarly, there are natural equivalences $CH_{S^{j-1}}^\bullet(A) \cong CH_{S^{j-1} \times D^{m-j}}^\bullet(A)$ of CDGAs. It follows, since $j_{k+m-j+1}^* (CH_{S^{j-1}}^\bullet(A)) \cong \int_{S^{j-1}} A$, that we get a natural equivalence

$$j_{k+1}^* (CH_{S^{j-1} \times D^{m-j}}^\bullet(A)) \cong i_{m-j}^* j_{k+m-j+1}^* (CH_{S^{j-1} \times D^{m-j}}^\bullet(A)) \cong \int_{S^{j-1} \times D^{m-j}} A$$

which finishes the proof in the compact case.

If M is non-compact, we still have a handle decomposition, but we may have to attach countable many handles to go from M_i to M_{i+1} . In particular, we can find an increasing sequence of relatively compact open subsets $M_i = X_0 \subset X_1 \subset \cdots \subset X_n \subset \cdots \subset \bigcup_{n \geq 0} X_n = M_{i+1}$ (for instance by choosing $\overline{X_n}$ to be the result of attaching the first n i -handles to M_i). We wish to prove the result by induction on i . Note first that, by definition of the Hochschild chain functor and Lemma 6, there is an equivalence (for the underlying E_k -algebras structures)

$$CH_{M_0}^\bullet(A) \cong \varinjlim_{\substack{F_0 \subset I_0 \\ F_0 \text{ finite}}} \left(\bigotimes_{f \in F_0} CH_{D^m}^\bullet(A) \right) \cong \varinjlim_{\substack{F_0 \subset I_0 \\ F_0 \text{ finite}}} \left(\bigotimes_{f \in F_0} \int_{D^m} (A) \right) \cong \int_{M_0} A$$

which proves the result for M_0 . Now, assume we have a natural equivalence $CH_{M_i}^\bullet(A) \cong \int_{M_i} A$. Writing $M_i = X_0 \subset X_1 \subset \cdots \subset X_n \subset \cdots \subset \bigcup_{n \geq 0} X_n = M_{i+1}$, by the above argument for the finite handles case, we have natural equivalences $CH_{X_n}^\bullet(A) \cong \int_{X_n} A$

for all n and thus a commutative diagram

$$\begin{array}{ccc}
 \varinjlim_{n \in \mathbb{N}} CH_{X_n}^\bullet(A) & \xrightarrow{\simeq} & \varinjlim_{n \in \mathbb{N}} \int_{X_n} A \\
 \downarrow & & \downarrow \\
 CH_{M_{i+1}}^\bullet(A) & \longrightarrow & \int_{M_{i+1}} A
 \end{array} \quad (19)$$

By Lemmas 7 and Lemma 1, the vertical arrows are equivalences, hence the lower map is too, which finishes the induction.

Remark 14. A geometric intuition behind Theorem 5 can be seen as follows. Let M be a dimension m manifold. Since a CDGA A is an E_n -algebra for any n , the topological chiral homology $\int_M A$ is defined for any n -framing of M , and is an E_{n-m} -algebra. Further, if M is n -framed (hence we have chosen a trivialization of $M \times D^{n-m}$), then M is also naturally $(n+k)$ -framed for any integer k . Since A is a CDGA, it is an E_{n+k} -algebra as well and thus we could have used the trivialization of $M \times D^{n-m} \times D^k \cong M \times D^{n+k-m}$ as well to define $\int_M A$ as an E_{n-m+k} -algebra.

It is well known that the transversality theorem implies that two embeddings $\phi_1 : M \rightarrow S^n$ and $\phi_2 : M \rightarrow S^n$ of M are isotopic if n is large enough. In particular, for large n the framing that comes from the embedding into S^n is unique. This unique invariant of M is called the stable normal bundle. Note that this implies that any two abstract framings of $M \times D^k$ and $M \times D^l$ are stably equivalent since, for example, for $M \times D^k$ we can make M sit inside \mathbb{R}^n and then the normal bundle of M in \mathbb{R}^n is the complement of the framing of the tangent bundle of $M \times D^k$ in $\mathbb{R}^n \times D^k$.

Building upon the last remark, we see that the topological chiral homology of an $(m+k)$ -framed manifold M with value in a CDGA should be equivalent (up to neglect of structure) to the topological chiral homology of M equipped with the stable normal framing so that we get

Corollary 12. *Topological chiral homology with values in CDGAs is independent of the framing. In other words, if M_1 and M_2 are diffeomorphic manifolds equipped respectively with an $(n+k_1)$ -framing and $(n+k_2)$ -framing, then there is a canonical equivalence $\int_{M_1} A \cong \int_{M_2} A$ of $E_{\min(k_1, k_2)}$ -algebras.*

Proof. By Theorem 5, there are natural equivalences $\int_{M_1} A \cong CH_{M_1}^\bullet(A)$ of E_{k_1} -algebras and $\int_{M_2} A \cong CH_{M_2}^\bullet(A)$ of E_{k_2} -algebras. Since M_1 and M_2 are diffeomorphic, we have an equivalence $CH_{M_1}^\bullet(A) \cong CH_{M_2}^\bullet(A)$ as CDGAs.

Let us conclude this section by mentioning briefly that Theorem 5 extends to not necessarily framed manifolds. Indeed, there is a version of topological chiral homology with values in *unoriented* E_n -algebras:

Definition 5. *The category of unoriented E_n -algebras, denoted $\mathbb{E}_n^{O(n)}\text{-Alg}_\infty$ is defined as the $((\infty, 1)\text{-})$ category of symmetric monoidal functors*

$$\mathbb{E}_n^{O(n)}\text{-Alg}_\infty := \text{Fun}^\otimes(\text{Disk}_n, k\text{-Mod}_\infty)$$

where Disk_n is the category with objects the integers and morphism the spaces $\text{Disk}_n(k, \ell) := \text{Emb}(\coprod_k \mathbb{R}^n, \coprod_\ell \mathbb{R}^n)$ of smooth embeddings of k disjoint copies of a disk \mathbb{R}^n into ℓ such copies; the monoidal structure is induced by disjoint union of copies of disks.

One can form the topological chiral homology $\int_M A$, see [11, Definition 3.15] (or [24, 25]) for any manifold M and unoriented E_n -algebra A .

Remark 15. Definition 5 above is extracted from [11, 24]. There is an natural action of the orthogonal group $O(n)$ on $E_n\text{-Alg}_\infty$, see [25]; the category $(E_n\text{-Alg}_\infty)^{hO(n)}$ of $O(n)$ -homotopy fixed points is equivalent to the category $\mathbb{E}_n^{O(n)}\text{-Alg}_\infty$ of Definition 5. In particular, any CDGA is an unoriented E_n -algebra.

If one replaces the action of $O(n)$ by $SO(n)$, one recovers the notion of *oriented* E_n -algebras which are commonly known as *framed* E_n -algebras in the literature.

A proof similar to the one of 5 yields

Proposition 12. *Let M be an n -dimensional manifold (non necessarily framed nor oriented) and A a CDGA, there is an natural equivalence $CH_M^\bullet(A) \cong \int_M A$ (of E_k -algebras if $M \cong N \times \mathbb{R}^k$ with N a dimension $n - k$ manifold).*

5.3. Topological chiral homology as a factorization algebra. In this section we give a precise relationship between factorization algebras, topological chiral homology for (stably) framed manifolds, and E_n -algebras.

5.3.1. Topological chiral homology and factorization algebras for n -framed manifolds. For any manifold M of dimension m which is n -framed (i.e. $M \times D^{n-m}$ is framed) and E_n -algebra A , we can consider the topological chiral homology $\int_M A$ as well as $\int_U A$ for every open subset U in M (equipped with the induced framing). Further, if U_1, \dots, U_k are pairwise disjoint open subsets of $V \in Op(M)$, there is a canonical equivalence ([24, Theorem 3.5.1])

$$\int_{U_1} A \otimes \cdots \otimes \int_{U_k} A \xrightarrow{\cong} \int_{U_1 \cup \dots \cup U_k} A$$

and a natural map $\int_{U_1 \cup \dots \cup U_k} A \rightarrow \int_V A$ (since any ball in $\bigcup U_i$ is a ball in V). Composing these two maps yield natural maps of E_{m-n} -algebras

$$\mu_{U_1, \dots, U_k, V} : \int_{U_1} A \otimes \cdots \otimes \int_{U_k} A \longrightarrow \int_V A. \quad (20)$$

Proposition 13. *Let M be an n -framed manifold of dimension m .*

1. *For any E_n -algebra A , the rule $U \mapsto \int_U A$ (for U open in M) together with the structure maps $\mu_{U_1, \dots, U_k, V}$ (20) define a locally constant factorization algebra $\mathcal{TC}_M(-, A)$ on M , such that $\mathcal{TC}_M(U, A) = \int_U A$ is canonically an E_{n-m} -algebra for any open U .*
2. *The rule $A \mapsto \mathcal{TC}_M(-, A)$ defines a functor $\mathcal{TC}_M : E_n\text{-Alg}_\infty \rightarrow \text{Fac}_M(E_{n-m}\text{-Alg}_\infty)$ which fits into the following commutative diagram*

$$\begin{array}{ccc} E_n\text{-Alg}_\infty & \xrightarrow{\int_M} & E_{n-m}\text{-Alg}_\infty \\ \mathcal{TC}_M \downarrow & \nearrow HF(-) & \\ \text{Fac}_M(E_{n-m}\text{-Alg}_\infty) & & \end{array}$$

In other words topological chiral homology computes the factorization homology of \mathcal{TC}_M .

The idea behind Proposition 13 is that for any submanifold U of M and E_n -algebra A , we can cover U by (a coherent family of) open balls on which A defines a locally constant factorization algebra. Gluing these data defines a factorization algebra on U whose homology can be computed from the balls by using the gluing/locality lemma given above (Lemma 5). Since the topological chiral homology is equivalent to A on balls and satisfy a similar locality axiom, they agree on U (by an argument similar to the proof of Theorem 5).

Proof (Proof of Proposition 13). For any open subset V of M , the topological chiral homology $\int_V A$ is the colimit $\varinjlim \psi_V$, where $\psi_V : N(\text{Disj}(V)) \rightarrow k\text{-Mod}_\infty$ is the

diagram given by the formula $\psi_V(V_1 \cup \dots \cup V_n) = \int_{V_1} A \otimes \dots \otimes \int_{V_n} A$ where V_1, \dots, V_n are disjoint open sets homeomorphic to a ball (Definition 4). In particular, the structure maps $\mu_{U_1, \dots, U_n, V}$ are induced by a map of colimits and it is easy to check that they are natural with respect to open embeddings and thus define a prefactorization algebra. Hence $\mathcal{TC}_M(-, A)$ is functorially (in A) a *prefactorization* algebra on M with value in E_{n-m} -algebras.

To prove that $\mathcal{TC}_M(-, A)$ is actually a *factorization* algebra, the idea is first to use a handle body decomposition to define another locally constant factorization algebra \mathcal{F}_M on M (whose factorization homology $HF(\mathcal{F}, M)$ is equivalent to $\int_M A$) and then to prove that this factorization algebra is equivalent to \mathcal{TC}_M . Note that by Lemma 4, if \mathcal{F}_M is a locally constant factorization algebra on M with value in E_d -algebras such that $\int_M A \cong HF(\mathcal{F}_M, M)$, then we have a natural factorization algebra $\mathcal{F}_{M \times \mathbb{R}^d}$ on $M \times \mathbb{R}^d$. Further $HF(\mathcal{F}_{M \times \mathbb{R}^d}, M \times \mathbb{R}^d) \cong \int_M A$ as an E_d -algebra.

We start with the case of open balls. By definition of topological chiral homology, for every manifold B homeomorphic to an m -dimensional ball, there is a natural equivalence $\int_B A \cong A$ of E_{n-m} -algebras (where the E_{n-m} -algebra structure of A is by restriction of structure), see [24, 25]. By a result of Lurie [24] (also see [9, Proposition 3.4.1] or Proposition 10), there is a locally constant factorization algebra \mathcal{F}_B on B whose factorization homology is isomorphic to A .

We need to prove that the (locally constant) factorization algebra \mathcal{F}_B is equivalent to \mathcal{TC}_B , i.e., that there are equivalences of prefactorization algebras $\mathcal{F}_B(U) \cong \int_U A$ for any open subset $U \subset B$. The proof is identical to the proof of Theorem 5. Indeed, by Lemma 5, the homology of \mathcal{F}_B satisfies the excision property and further convert disjoint union to tensor products. So that we can use the handle decomposition argument of Theorem 5 to prove that $\mathcal{TC}_B \cong \mathcal{F}_B$ and thus that \mathcal{TC}_B is a factorization algebra.

If $M = \coprod_{i=1}^l B_i$ is a disjoint union of balls of dimension m , then we also deduce that \mathcal{TC}_M is a factorization algebra since $\int_{U_1} A \otimes \dots \otimes \int_{U_l} A \cong \int_{U_1 \sqcup \dots \sqcup U_l} A$ for any open subsets $U_i \subset B_i$. In particular, this applies to the case of $S^0 \times D^d$ ($d \geq 0$). Now let $M \cong S^m \times D^d$ ($d \geq 0$, $m \geq 1$) be framed (we do not assume it is embedded as an open set of an euclidean space). We work by induction on m so that we may assume the result of the proposition is known for $S^{m-1} \times D^l$.

Assume we have a cover $U \cup V$ of a space X and factorization algebras $\mathcal{B}_U, \mathcal{B}_V, \mathcal{B}_{U \cap V}$ on U, V and $U \cap V$ with equivalences $\mathcal{B}_{U \cap V} \xrightarrow{\sim} \mathcal{B}_{U|U \cap V}$ and $\mathcal{B}_{U \cap V} \xrightarrow{\sim} \mathcal{B}_{V|U \cap V}$. Then we can glue these factorization algebras to define a factorization algebra on X , see [8] (note that this descent property can be generalized to arbitrary covers). We wish to apply this to a decomposition of the m -sphere S^m as $S^m \cong D^+ \cup_{S^{m-1} \times D^1} D^-$ where

D^+ and D^m are homeomorphic to framed open balls. By the above analysis, there are locally constant factorization algebras \mathcal{F}_+ , \mathcal{F}_- on $D^+ \times D^d$ and $D^- \times D^d$ which are equivalent to $\mathcal{TC}_{D^{m+d}}(A)$. Restricting these equivalences to $S^{m-1} \times D^1 \times D^d$, we get an equivalence

$$\mathcal{F}_+|_{S^{m-1} \times D^1 \times D^d} \xrightarrow{\cong} \mathcal{F}_-|_{S^{m-1} \times D^1 \times D^d}.$$

Since $S^{m-1} \times D^1 \times D^d$ is the intersection of $D^+ \times D^d$ with $D^- \times D^d$, we thus get a locally constant factorization algebra \mathcal{F} on their union $S^m \times D^d$. It follows from Lemma 5 and Proposition 11, that $\mathcal{F}(S^m \times D^d) \cong \int_{S^m \times D^d} A$ as an E_d -algebra. The equivalences $\mathcal{F}(U) \cong \int_U A$ follows for any open proper subset of $S^m \times D^d$ as in the proof of Theorem 5 (using again the excision property of chiral homology and factorization algebra homology). It follows that $\mathcal{TC}_{S^m \times D^d}$ is a factorization algebra.

The case of general n -framed manifolds M is done similarly. Using a handle decomposition, induction, and the descent property of factorization algebras, we build a locally constant factorization algebra \mathcal{F}_M on M and then prove, as in the proof of Theorem 5, that, for any open U , $\mathcal{F}_M(U) \cong \int_U M$.

To finish the proof, note that the factorization algebra $\mathcal{TC}_M(-, A)$ is locally constant by construction (since $\int_B A \cong A$ for any ball B) and its factorization homology is precisely the topological chiral homology of M with value in A . Further $\mathcal{TC}_M(-, A)$ is functorial in A since topological chiral homology is.

Remark 16. Theorem 5 follows easily from Proposition 13 and Corollary 9.

5.3.2. Topological chiral homology and factorization algebras for $\mathbb{E}_n[M]$ -algebras. We now go beyond the notion of n -framed manifolds M , and, more generally, consider locally constant algebras over an operad $\mathbb{E}_n[M]$, for which there might not exist a globally defined E_n -algebra.

Following Lurie [24], topological chiral homology can also be defined for a (locally constant) family of E_n -algebras parametrized by the points in $M \times D^{m-n}$ even if M is not n -framed. Such objects are (locally constant) algebras over an (∞) -operad $\mathbb{E}_n[M] := \mathbb{E}_{M \times D^{n-m}}^\otimes$, the operad of little n -cubes in $M \times D^{m-n}$, see [24, Definition 5.2.4.1] (here M is still of dimension m , and of course one can choose $m = n$). By [24, Theorem 5.2.4.9], we can also describe an $\mathbb{E}_n[M]$ -algebra as a locally constant $N(\text{Disk}(M \times D^{n-m}))$ -algebra. Indeed, by [24, Remark 5.3.2.7] we can extend Definition 4 to an $\mathbb{E}_n[M]$ -algebra \mathcal{A} as well by replacing the last equivalence in (13) by $\mathcal{A}(V_1) \otimes \cdots \otimes \mathcal{A}(V_n)$. That is, there is an equivalence

$$\int_M \mathcal{A} \cong \varinjlim \int_{V_1} A \otimes \cdots \otimes \int_{V_n} A \cong \varinjlim \mathcal{A}(V_1) \otimes \cdots \otimes \mathcal{A}(V_n) \quad (21)$$

where V_1, \dots, V_n are disjoint open sets homeomorphic to a ball.

Lemma 8 (Lurie [24]). *Let M be a manifold and \mathcal{A} be an $\mathbb{E}[M]$ -algebra. Assume that there is a codimension 1 submanifold N of M with a trivialization $N \times D^1$ of its neighborhood such that M is decomposable as $M = X \cup_{N \times D^1} Y$ where X, Y are submanifolds of M glued along $N \times D^1$. Then*

1. $\int_{N \times D^1} \mathcal{A}$ is an E_1 -algebra and $\int_X \mathcal{A}$ and $\int_Y \mathcal{A}$ are right and left modules over $\int_{N \times D^1} \mathcal{A}$.

2. The natural map

$$\int_X \mathcal{A} \otimes_{\int_{N \times D^1} \mathcal{A}}^{\mathbb{L}} \int_Y \mathcal{A} \longrightarrow \int_M \mathcal{A}$$

is an equivalence.

Proof. The lemma is explained in [24] after Theorem 5.3.4.14. A detailed proof⁶ is given in [11, Proposition 3.27] (as well as in [2, Section 3.5]).

The generalization of Proposition 13 to $\mathbb{E}_n[M]$ -algebras is:

Theorem 6. *Let M be a manifold of dimension m and $d \in \mathbb{N}$ an integer.*

1. *The rule $\mathcal{A} \mapsto \left(U \mapsto \int_U \mathcal{A} \right)$ defines a functor of $(\infty, 1)$ -algebras $\mathcal{TC}_M : \mathbb{E}_d[M]\text{-Alg} \rightarrow \text{Fac}_M^{lc}(E_d\text{-Alg})$ which fits into a commutative diagram*

$$\begin{array}{ccc} \mathbb{E}_d[M]\text{-Alg} & \xrightarrow{\int_M} & E_d\text{-Alg}_\infty \\ \mathcal{TC}_M \downarrow & \nearrow HF(-) & \\ \text{Fac}_M^{lc}(E_d\text{-Alg}_\infty) & & \end{array}$$

2. *The functor $\mathcal{TC}_M : \mathbb{E}_d[M]\text{-Alg} \rightarrow \text{Fac}_M^{lc}(E_d\text{-Alg})$ is an equivalence of $(\infty, 1)$ -categories.*

In particular, any locally constant factorization algebra \mathcal{F} on M with values in E_d -algebras is equivalent to $\mathcal{TC}_M(\mathcal{A})$ for a unique (up to equivalences) $\mathbb{E}_d[M]$ -algebra \mathcal{A} , i.e., algebra over the operad of little cubes in $M \times D^d$. Further, topological chiral homology of an open set U with value in the associated $\mathbb{E}_{M \times D^d}^\otimes$ -algebra computes the (derived) sections of the factorization algebra.

Proof. We first deal with assertion (1). By Lemma 4, it is enough to prove that the rule $U \mapsto \int_U \mathcal{A}$, together with the structure maps

$$\int_{U_1} \mathcal{A} \otimes \cdots \otimes \int_{U_\ell} \mathcal{A} \xrightarrow{\sim} \int_{U_1 \cup \cdots \cup U_\ell} \mathcal{A} \longrightarrow \int_V \mathcal{A} \quad (22)$$

for U_i 's pairwise disjoint open subsets of $V \in \text{Op}(M \times D^d)$, defines a locally constant factorization algebra on $M \times D^d$, naturally in $\mathcal{A} \in \mathbb{E}_{M \times D^d}^\otimes\text{-Alg}$. Note that, by [24, Theorem 5.2.4.9], the $\mathbb{E}_{M \times D^d}^\otimes$ -algebra \mathcal{A} satisfies that, for any ball B which is a subset of a ball B' , the canonical map $\int_B \mathcal{A} \cong \mathcal{A}(B) \rightarrow \mathcal{A}(B') \cong \int_{B'} \mathcal{A}$ is an equivalence in $k\text{-Mod}_\infty$. Now we can apply the same proof as the one of Proposition 13 with \mathcal{A} instead of A , using Lemma 8 instead of Proposition 11.

We now prove assertion (2). By [24, Theorem 5.2.4.9], the canonical embedding $\theta : \mathbb{E}_{M \times D^d}^\otimes\text{-Alg} \rightarrow N(\text{Disk}(M \times \mathbb{R}^d))\text{-Alg}$ induces an equivalence between $\mathbb{E}_{M \times D^d}^\otimes\text{-Alg}$ and locally constant $N(\text{Disk}(M \times \mathbb{R}^d))\text{-Alg}$; we write $N(\text{Disk}^{lc}(M))\text{-Alg}$ for the

⁶ Note that $\mathbb{E}[M]$ -algebras are denoted \mathcal{E}_M -algebras in [11] and \mathbf{Disk}_n^M -algebras in [12]; here the map $M \rightarrow B\text{Top}(n)$ is given by the tangent bundle of M .

latter subcategory. It is thus enough to define a functor $\mathcal{E}\mathcal{A}_M : \text{Fac}_M^{\text{lc}}(E_d\text{-Alg}_\infty) \rightarrow N(\text{Disk}^{\text{lc}}(M))\text{-Alg}$ such that $\mathcal{TC}_M \circ \mathcal{E}\mathcal{A}_M$ and $\mathcal{E}\mathcal{A}_M \circ \mathcal{TC}_M$ are respectively equivalent to the identity functors of $\text{Fac}_M^{\text{lc}}(E_d\text{-Alg}_\infty)$ and $N(\text{Disk}^{\text{lc}}(M))\text{-Alg}$. Let \mathcal{F} be in $\text{Fac}_M^{\text{lc}}(E_d\text{-Alg}_\infty)$. By Lemma 4, we can think of \mathcal{F} as a locally constant factorization algebra on $M \times D^d$. Let $B \in \text{Op}(M \times D^d)$ be homeomorphic to a ball. Then the restriction $\mathcal{F}|_B$ is a locally constant factorization algebra on $B \cong \mathbb{R}^n$, thus is equivalent to an E_n -algebra (which is canonically equivalent to $\mathcal{F}(B)$). Further, for any finite family B_1, \dots, B_ℓ of pairwise disjoint open subsets homeomorphic to a ball and U an open subset homeomorphic to a ball containing the B_i 's, the locally constant factorization algebra structure defines a canonical map

$$\gamma_{B_1, \dots, B_\ell, U} : \mathcal{F}(B_1) \otimes \dots \otimes \mathcal{F}(B_\ell) \longrightarrow \mathcal{F}(U)$$

which is an equivalence if $\ell = 1$. The maps $\gamma_{B_1, \dots, B_\ell, U}$ are compatible in a natural way. This shows that the collection $\mathcal{F}(B)$ for all open sets $B \subset M \times D^d$ homeomorphic to a ball is a locally constant $N(\text{Disk}(M \times \mathbb{R}^d))$ -algebra, denoted $\mathcal{A}_{\mathcal{F}}$ and we define the functor $\mathcal{E}\mathcal{A}_M$ to be defined by $\mathcal{E}\mathcal{A}_M(\mathcal{F}) := \mathcal{A}_{\mathcal{F}}$ (for $M = pt$ it is the same as the functor f_{or} defined in the proof of Proposition 10). In other words, the functor $\mathcal{E}\mathcal{A}_M(\mathcal{F})$ is simply induced by the composition $\text{Disk}(M \times \mathbb{R}^d) \hookrightarrow \text{Op}(M \times \mathbb{R}^d) \xrightarrow{\mathcal{F}} k\text{-Mod}_\infty$. By abuse of notation, we also write $\mathcal{A}_{\mathcal{F}}$ for the associated (well defined up to equivalences) $\mathbb{E}_{M \times D^d}^\otimes$ -algebra.

Let \mathcal{A} be an $\mathbb{E}_{M \times D^d}^\otimes$ -algebra. By construction $\mathcal{E}\mathcal{A}_M \circ \mathcal{TC}_M(\mathcal{A})$ is the (locally constant) $N(\text{Disk}(M \times \mathbb{R}^d))$ -algebra given, on any (open set homeomorphic to an euclidean) ball B , by

$$\left(\mathcal{E}\mathcal{A}_M \circ \mathcal{TC}_M(\mathcal{A}) \right)(B) = \int_B \mathcal{A} \cong \mathcal{A}(B)$$

by definition of topological chiral homology [24, Example 5.3.2.8]. Hence there is a canonical equivalence $\mathcal{E}\mathcal{A}_M \circ \mathcal{TC}_M(\mathcal{A}) \cong \mathcal{A}$ of locally constant $N(\text{Disk}(M \times \mathbb{R}^d))$ -algebras and thus of $\mathbb{E}_{M \times D^d}^\otimes$ -algebras as well.

It remains to prove that, there are natural equivalences $\mathcal{TC}_M(\mathcal{A}_{\mathcal{F}}) \cong \mathcal{F}$ of factorization algebras, where $\mathcal{A}_{\mathcal{F}}$ is the $\mathbb{E}_{M \times D^d}^\otimes$ -algebra $\mathcal{E}\mathcal{A}_M(\mathcal{F})$ associated to \mathcal{F} as above. Fixing a Riemannian metric on $M \times D^d$, we can find a cover $\text{Ball}^g(M \times D^d)$ of M given by open sets in $M \times D^d$ which are geodesically convex. On every $U \in \text{Ball}^g(M \times \mathbb{R}^d)$, the restrictions $\mathcal{TC}|_U(\mathcal{A}_{\mathcal{F}})$ and $\mathcal{F}|_U$ are naturally isomorphic by the above paragraph. In particular, for any set $U_I := \bigcap_{i \in I} U_i$ and any subset $J \subset I$, the following diagram

$$\begin{array}{ccc} \mathcal{TC}|_{U_I}(\mathcal{A}_{\mathcal{F}}) & \xrightarrow{\cong} & \mathcal{F}|_{U_I} \\ \downarrow & & \downarrow \\ \mathcal{TC}|_{U_I \setminus J}(\mathcal{A}_{\mathcal{F}}) & \xrightarrow{\cong} & \mathcal{F}|_{U_I \setminus J} \end{array}$$

is commutative. Since $\mathcal{TC}_M(\mathcal{A}_{\mathcal{F}})$ and \mathcal{F} are the factorization algebras obtained by descent from their restrictions on the cover $\text{Ball}^g(M \times \mathbb{R}^d)$, on which they are naturally equivalent, it follows that \mathcal{F} is equivalent to $\mathcal{TC}_M(\mathcal{A}_{\mathcal{F}})$.

Example 12. Since $S^2 \times D^1$ embeds as an open set in \mathbb{R}^3 , any E_3 -algebra A yields, by restriction, a (locally constant) factorization algebra \mathcal{A}_{S^2} on S^2 (with values in $E_1\text{-Alg}$) (Lemma 4). By Theorem 6 and Proposition 11, decomposing the sphere as two disks glued along the equator, we get that the factorization homology of \mathcal{A}_{S^2} is given by

$$HF(\mathcal{A}_{S^2}) \cong A \underset{CH_{S^1}(A)}{\overset{\mathbb{L}}{\otimes}} A$$

as an E_1 -algebra. Here $CH_{S^1}(A)$ is the usual Hochschild chain complex of the (underlying) E_1 -algebra structure of A , which is naturally an E_2 -algebra by [24, Theorem 5.3.3.11] and Proposition 11.

Similarly, any E_2 -algebra B yields a (translation invariant and locally constant) factorization algebra on \mathbb{R}^2 , and thus a (locally constant) factorization algebra \mathcal{B}_T on a torus $T = S^1 \times S^1 \cong \mathbb{R}^2/\mathbb{Z}^2$. Since T is framed, we can also define its topological chiral homology directly using the framing. It follows easily from the uniqueness statement in Theorem 6, that \mathcal{B}_T is equivalent to $\mathcal{TC}_T(B)$ in $\text{Fac}_T(k\text{-Mod}_\infty)$.

Note that the two canonical projections $p_1, p_2 : \mathbb{R}^2 \rightarrow \mathbb{R}$ define two locally constant factorization algebras $p_{1*}(\mathcal{B}), p_{2*}(\mathcal{B})$ on \mathbb{R} and thus, two E_1 -algebras B_1 and B_2 . Now, cutting the torus along two meridian circles, we get two copies of $S^1 \times D^1 \cong \mathbb{R}^2/(\mathbb{Z} \oplus \{0\})$ glued along their boundaries. By Theorem 6 again, the topological chiral homology of $S^1 \times D^1$ is the same as the factorization algebra homology of the descent factorization algebra $\mathcal{B}^{\mathbb{Z} \oplus \{0\}}$. Thus $\int_{S^1 \times D^1} B$ is equivalent to the usual Hochschild chain complex $CH_{S^1}(B_1)$ and the later complex inherits an E_1 -structure from the E_2 -algebra structure of \mathcal{B} . From Proposition 11, we deduce a natural equivalence (in $k\text{-Mod}_\infty$)

$$HF(\mathcal{B}_T) \cong CH_{S^1}(B_1) \underset{CH_{S^1}(B_1) \otimes (CH_{S^1}(B_1))^{\text{op}}}{\overset{\mathbb{L}}{\otimes}} CH_{S^1}(B^1) \cong CH_{S^1}(CH_{S^1}(B_1)).$$

Note that if B was actually a CDGA, then the later equivalence follows directly from Corollary 11.

5.4. Some applications.

5.4.1. Another construction of topological chiral homology for framed manifolds. Let M be an m -dimensional manifold which is n -framed. Given an E_n -algebra A , we can define the topological chiral homology $\int_M A$ of M with values in A . By Proposition 13 and Theorem 6, $\int_M A$ is the factorization homology of a factorization algebra on $M \times D^{n-m}$. We explain how to construct this factorization algebra directly.

Since M is n -framed, there is a bundle isomorphism $\varphi : T(M \times D^{n-m}) \xrightarrow{\sim} \mathbb{R}^n$ where \mathbb{R}^n is a trivial bundle over $M \times D^{n-m}$. Choosing a Riemannian metric on $M \times D^{n-m}$, we have, using the spray associated to the exponential map, canonical diffeomorphisms of (a basis of) open neighborhoods of any $x \in M \times D^{n-m}$ to open sets in the tangent space $T(M \times D^{n-m})_x$ of $M \times D^{n-m}$ at x . Composing with the map φ induced by the framing, we get diffeomorphisms $U \mapsto \psi(U_x) \in \text{Op}(\mathbb{R}^n)$ where U is a (geodesically convex) open neighborhood of x .

Let \mathcal{U} be the cover of $M \times D^{n-m}$ obtained by considering the U above such that $\phi_x(U_x) \in \text{Ball}(\mathbb{R}^n)$ is an euclidean ball. The cover \mathcal{U} is a factorizing basis of open subsets of $M \times D^{n-m}$.

To any $U \in \mathcal{U}$, we associate $\mathcal{A}(U) = A$, a (fixed) E_n -algebra. We wish to extend \mathcal{A} into a factorization algebra. Since A is an E_n -algebra, it defines a locally constant factorization algebra on \mathbb{R}^n (see [9, 24] and Proposition 10), which we, by abuse of notation, again denote by A . For any pairwise disjoint $U_1, \dots, U_n \in \mathcal{U}$, and $V \in \mathcal{U}$ such that $U_i \subset V$ ($i = 1 \dots n$), we define the structure maps $\mu_{U_1, \dots, U_n, V}$ (see Sect. 4.2) by the following commutative diagram:

$$\begin{array}{ccc} \mathcal{A}(U_1) \otimes \dots \otimes \mathcal{A}(U_n) & \xrightarrow{\mu_{U_1, \dots, U_n, V}} & \mathcal{A}(V) \\ \simeq \downarrow & & \downarrow \simeq \\ A(\psi(U_1)) \otimes \dots \otimes A(\psi(U_n)) & \longrightarrow & A(\psi(V)) \end{array}$$

where the lower arrow is given by the E_n -algebra structure of A . This yields a \mathcal{U} -factorization algebra (in the sense of [8] and Sect. 4.2) since A is a factorization algebra on \mathbb{R}^n and $M \times D^{n-m}$ is framed. By [8, Section 3], we can now extend \mathcal{A} to a factorization algebra on $M \times D^{n-m}$.

Corollary 13. *There is an equivalence of E_{n-m} -algebras*

$$\int_M A \cong HF(M, \mathcal{A}).$$

Proof. For any ball U , we have a natural equivalence $\int_U A \cong A \cong \mathcal{A}(U)$. Now the result follows from Theorem 6 (and its proof) after taking global sections.

Note that topological chiral homology $\mathcal{TC}_M(A)$ is independent of the Riemannian metric, hence the factorization algebra $\mathcal{A} \in \text{Fac}_{M \times D^{n-m}}^{lc}(k\text{-Mod}_\infty)$ on M thus obtained is also independent of the Riemannian metric.

5.4.2. Interpretation of topological chiral and higher Hochschild in terms of mapping spaces. As we have already noticed, higher Hochschild chains behave much like mapping spaces (and thus so do $\int_M A$ for CDGAs A). Indeed,

Corollary 14. *Let $A = \Omega^*N$ be the de Rham forms on a d -connected manifold (with its usual differential). Then for any manifold M of dimension $m \leq d$, there is a natural quasi-isomorphism $\int_M A \cong \Omega^*(N^M)$, the space of (Chen) de Rham forms of the mapping space $N^M = \text{Map}(M, N)$.*

In other words, topological chiral homology of M with value in Ω^*N calculates the mapping space N^M (if N is sufficiently connected).

Proof. By Theorem 5, we are left to a similar statement for $CH_M(\Omega^*N)$. Since M is m -dimensional it has a simplicial model with no non-degenerate simplices in dimensions above m . Now the result follows from [14, Proposition 2.5.3 and Proposition 2.4.6].

Remark 17. By [14, Section 2.4], there is a canonical map $\int_M \Omega^*N \rightarrow \Omega^*(N^M)$. Further, it is possible to replace N by any nilpotent space (by mimicking the proof of [14, Propositions 2.5.3 and 2.4.6]) and Ω^*N by a Sullivan model of N .

We now give a (derived/homotopical) algebraic geometry statement. Recall that k denotes a field of characteristic zero and let \mathbf{dSt}_k be the (model) category of *derived stacks* over k described in details in [30, Section 2.2] (which is a derived enhancement of the category of stacks over k). This category admits internal Hom's that we denote by $\mathbb{R} \operatorname{Map}(\mathfrak{X}, \mathfrak{Y})$ following [30, 31]. To any simplicial set X_\bullet , we associate the constant simplicial presheaf $k\text{-Alg} \rightarrow s\text{Set}$ defined by $R \mapsto X_\bullet$ and we denote \mathfrak{X} the associated stack. For a (derived) stack \mathfrak{Y} , we denote $\mathcal{O}_{\mathfrak{Y}}$ its functions [30] (i.e., $\mathcal{O}_{\mathfrak{Y}} := \mathbb{R} \operatorname{Hom}(\mathfrak{Y}, \mathbb{A}^1)$).

Corollary 15. *Let $\mathfrak{X} = \mathbb{R} \operatorname{Spec}(R)$ be an affine derived stack (for instance an affine stack) [30]. Then the Hochschild chains over X_\bullet with coefficient in R represent the mapping stack $\mathbb{R} \operatorname{Map}(\mathfrak{X}, \mathfrak{X})$. That is,*

$$\mathcal{O}_{\mathbb{R} \operatorname{Map}(\mathfrak{X}, \mathfrak{X})} \cong CH_{X_\bullet}^\bullet(R).$$

Proof. The bifunctor $(\mathfrak{X}, \mathbb{R} \operatorname{Spec}(R)) \mapsto \mathbb{R} \operatorname{Map}(\mathfrak{X}, \mathbb{R} \operatorname{Spec}(R))$ is contravariant in \mathfrak{X} and $R \in CDGA^{\leq 0}$. Thus, $\mathcal{O}_{\mathbb{R} \operatorname{Map}(\mathfrak{X}, \mathfrak{X})}$ defines a covariant bifunctor. Since $\mathbb{R} \operatorname{Map}(-, \mathfrak{X})$ sends (homotopy) limits to (homotopy) colimits, it follows from Theorem 2 (also see Remark 4) that $\mathcal{O}_{\mathbb{R} \operatorname{Map}(\mathfrak{X}, \mathfrak{X})}$ is equivalent to $CH_{X_\bullet}^\bullet(R)$.

Example 13. Let $B_\bullet \mathbb{Z}$ be the nerve of \mathbb{Z} and $\mathfrak{B}\mathbb{Z}$ its associated stack. Recall that there is an homotopy equivalence $S^1 \rightarrow |B_\bullet \mathbb{Z}|$ (actually induced by a simplicial set map, see [21]). From Corollary 15 we recover that the derived loop stack $L\mathfrak{X} := \mathbb{R} \operatorname{Map}(\mathfrak{B}\mathbb{Z}, \mathfrak{X})$ is represented by $CH_{B_\bullet \mathbb{Z}}^\bullet(R) \xrightarrow{\sim} CH_{S^1}^\bullet(R)$ the standard Hochschild chain complex of R as was proved in [31]. Similarly, the derived torus mapping stack $\mathbb{R} \operatorname{Map}(\mathfrak{B}\mathbb{Z} \times \mathfrak{B}\mathbb{Z}, \mathfrak{X})$ is represented by $CH_{S^1 \times S^1}^\bullet(R)$ and the secondary cyclic homology in the sense of [31] is represented by the homotopy fixed points $CH_{S^1 \times S^1}^\bullet(R)^{h(S^1 \times S^1)}$ with respect to the induced action of the simplicial group $B_\bullet \mathbb{Z} \times B_\bullet \mathbb{Z}$ on the derived mapping space.

Remark 18. Sheafifying (or rather stackifying) the higher Hochschild derived functor, it seems possible to extend Corollary 15 to general derived schemes.

5.4.3. Topological chiral Homology and homology spheres Topological chiral homology of CDGAs is a homology invariant, and thus, in particular, a homotopy invariant. Indeed, we have the following corollary.

Corollary 16. *Let $f : M \rightarrow N$ be smooth map between two manifolds inducing isomorphisms on homology and A be a CDGA, then $\int_M A \cong \int_N A$.*

Proof. This follows from Theorem 5 and the quasi-isomorphism invariance of $CH_{(-)}^\bullet(A)$ (Proposition 4).

Example 14. The composition $S^3 \rightarrow SO(3) \rightarrow SO(3)/I$, where I is the icosahedral group, induces an isomorphism on homology. To see this, note that the fundamental group of $SO(3)/I$ is the binary icosahedral group \tilde{I} which is a perfect group and therefore $H_1(SO(3)/I) = 0$. The result $SO(3)/I$ is the Poincaré homology sphere and has thus the same topological chiral homology with value in any CDGA as S^3 .

Remark 19. Note that we study topological chiral homology in the framework of chain complexes, i.e. we have fixed the $(\infty, 1)$ -category of chain complexes as our “ground” monoidal $(\infty, 1)$ -category. If one works in some other framework (such as topological spaces), one can expect to have more refined invariants.

Remark 20. Note that S^1 has two diffeomorphic 1-framings (specified by a choice of orientation). This accounts for the fact that classically there is only one Hochschild complex for associative algebras. There are countably many 2-framings for the circle, one for each integer, giving rise to equivalent topological chiral homologies when the integrand is a CDGA. It would therefore be meaningful to look for an explicit E_2 -algebra that distinguishes these framings from one another, if such exists. Similarly, it would be interesting to find an explicit E_3 -algebra that distinguishes the two 3-framings of S^1 .

5.4.4. Fubini formula for topological chiral homology The exponential law for Hochschild chains (Proposition 5) has an analogue for topological chiral homology.

Corollary 17. *Let M, N be manifolds and \mathcal{A} be an $E_d[M \times N]$ -algebra. Then, $\int_N \mathcal{A}$ has a canonical lift as an $E_d[M]$ -algebra and further, there is an equivalence of E_d -algebras*

$$\int_{M \times N} \mathcal{A} \cong \int_M \left(\int_N \mathcal{A} \right).$$

Proof. Replacing M by $M \times \mathbb{R}^d$ and using Lemma 4, it is enough to prove the result for $d = 0$. Since the homology of a factorization algebra on X is given by the pushforward along the canonical map $p : X \rightarrow pt$, by Theorem 6, one has

$$\int_{M \times N} \mathcal{A} \cong p_*(\mathcal{TC}_{M \times N}(\mathcal{A})) \cong p_*(\pi_*(\mathcal{TC}_{M \times N}(\mathcal{A}))) \quad (23)$$

where $\pi : M \times N \rightarrow M$ is the canonical projection. Since $\mathcal{TC}_{M \times N}(\mathcal{A})$ is locally constant, $\pi_*(\mathcal{TC}_{M \times N}(\mathcal{A}))$ is also locally constant whose value on an open ball $D \subset M$ is given by $\pi_*(\mathcal{TC}_{M \times N}(\mathcal{A})(U) \cong \mathcal{TC}_{M \times N}(\mathcal{A})(U \times N) \cong \int_N \mathcal{A}$. By Theorem 6, this defines the canonical $E_d[M]$ -algebra structure on $\int_N \mathcal{A}$ and the result now follows from the equivalence (23).

Example 15. Let M, N be $m + k$ -framed and $n + \ell$ -framed manifolds of respective dimension m, n and A be an $E_{m+n+k+\ell}$ -algebra. Then, Corollary 17 yields an equivalence of $E_{k+\ell}$ -algebras:

$$\int_{M \times N} A \cong \int_M \left(\int_N A \right).$$

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