HIGHER HOCHSCHILD COHOMOLOGY, BRANE TOPOLOGY AND CENTRALIZERS OF E_n-ALGEBRA MAPS

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ABSTRACT. We use factorization homology and higher Hochschild (co)chains to study various problems in algebraic topology and homotopical algebra, notably brane topology, centralizers of E_n-algebras maps and iterated bar constructions. In particular, we obtain an E_{n+1}-algebra model on the shifted integral chains $C_{\bullet+m}(Map(S^n, M))$ of the mapping space of the n-sphere into an m-dimensional orientable closed manifold M. We construct and use E_∞-Poincaré duality to identify the higher Hochschild cochains, modeled over the n-sphere, with the chains on the above mapping space, and then relate the Hochschild cochains to the deformation complex of the E_∞-algebra $C^*(M)$, thought of as an E_n-algebra. We invoke (and prove) the higher Deligne conjecture to furnish E_n-Hochschild cohomology, and all that is naturally equivalent to it, with an E_{n+1}-algebra structure and further prove that this construction recovers the sphere product. In fact, our approach to the Deligne conjecture is based on an explicit description of the E_n-centralizers of a map of E_∞-algebras $f : A \to B$ by relating it to the algebraic structure on Hochschild cochains modeled over spheres, which is of independent interest and explicit. More generally, we give a factorization algebra model/description of the centralizer of any E_n-algebra map and a solution of Deligne conjecture. We also apply similar ideas to the iterated bar construction. We obtain factorization algebra models for (iterated) bar construction of augmented E_m-algebras together with their E_n-coalgebras and E_{m-n}-algebra structures, and discuss some of its features. For E_∞-algebras we obtain a higher Hochschild chain model, which is an E_n-coalgebra. In particular, considering the E_∞-algebra structure of an n-connected topological space Y, we obtain a higher Hochschild cochain model of the natural E_n-algebra structure of the chains of the iterated loop space $C_*(\Omega^n Y)$.

CONTENTS

1. Introduction

The main goal of this paper is to apply the recent tools given by *factorization algebras* and *factorization homology* (or higher Hochschild (co)homology) to study various problems in algebraic topology and homological algebra, including the study of string and *brane topology*, existence and explicit description of *centralizers* of maps, which gives rise to a solution of higher Deligne conjecture, and the study of *iterated bar constructions* for (homotopically commutative) algebras and *iterated loop spaces*. These applications are the core of the sections 6, 7 and 8.

We start demonstrating these ideas by first explaining the starting point of our work. Our original motivation was the study of brane topology, as emphasized by D. Sullivan: the algebraic structure of the chains on the mapping space of the n-sphere into an orientable m-dimensional manifold M; the coefficient of the chains being over a field of arbitrary characteristic, or over the integers. The algebraic structure of the chains on the mapping spaces of spheres into a manifold has drawn considerable interest, following the work of Chas and Sullivan [CS] on the free loop space. It is now standard that the homology of the free loop space $LM = Map(S^1, M)$, shifted by the dimension of M, has an intriguing structure of a BV-algebra, and in particular of a Gerstenhaber algebra[1], that is of a (graded) commutative algebra endowed with a degree 1 Lie bracket satisfying the Leibniz rule. This structure

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[1]also called 1-Poisson algebra

is in fact part of a 2-dimensional homological conformal field theory (for instance see [**?**, BGNX, Lu4])); the BV-algebra structure comes from the genus 0 part of this topological conformal field theory.

Higher string topology, also referred to as *brane topology*, is a generalization of string topology in which the circle is replaced by the n-dimensional sphere. Sullivan and Voronov (see [CV]) have stated[2] that the (shifted) homology of the mapping sphere $Map(S^n, M)$ has the structure of a BV_n-algebra and in particular of an n-Poisson algebra (or n-braid algebra in the terminology of [KM]). The latter structure is the analogue of a Gerstenhaber algebra in which the Lie bracket is of degree n. A BV_n-algebra is an algebra over the homology of the operad of framed n-dimensional little disks, while an n-Poisson algebra is an algebra over the homology of the *little n-dimensional disks operad* (for instance see [CV, SW]); algebras over (the chains on) the little n-dimensional disks operad are usually called E_n-algebras.

The E_n-*algebras* form a hierarchy of homotopy commutative structures, whose commutativity increase with n, with E_1-algebras being essentially equivalent to dg-associative algebras. In particular, an E_2-algebra is a dg-associative algebra, with product $\cup_0$, together with an homotopy operator $\cup_1$ for the commutativity of the product $\cup_0$. Similarly, in an E_n-algebra, $\cup_1$ is homotopy commutative, the homotopy being given by an operator $\cup_2$ which is homotopy commutative and so on until an homotopy operator $\cup_{n-1}$ which is not (required to be) homotopy commutative. It is well known that the homology of an E_n-algebra is an n-Poisson algebra. These algebras are nowadays of fundamental importance in quantization (for instance see [KS, PTVV]).

Sullivan-Voronov's work leads to the following:

Question: *Is it possible to lift the n-Poisson algebra structure on the homology of $Map(S^n, M)$ to a structure of (framed) E_n-algebras on the (suitably shifted) chains of $Map(S^n, M)$ with coefficient in an arbitrary ring k?*

For an n-connected closed and oriented manifold M, we give a positive answer to this conjecture.

Theorem 7.1. *Let M be an n-connected Poincaré duality space whose homology groups are projective k-modules. The shifted chain complex $C_{*+\dim(M)}(Map(S^n, M))$ has a natural E_{n+1}-algebra structure which induces the Sullivan-Voronov sphere product in homology*

$$H_p\big(Map(S^n, M)\big) \otimes H_q\big(Map(S^n, M)\big) \to H_{p+q-\dim(M)}\big(Map(S^n, M)\big),$$

when M is an oriented closed manifold.

This E_{n+1}-algebra structure can be seen as a higher dimensional analogue of the genus 0 part of a topological conformal field theory.

Our approach is based on an *algebraic model* of the chains on the mapping spaces generalizing *Hochschild cochains*, a fruitful model for string topology operations. This algebraic model is an instance of factorization homology for commutative or E_∞-algebras which we develop in sections 3 and 4.

Hochschild cohomology groups of an associative algebra A with value in a bi-module N are defined as

$$HH^n(A, N) \cong H^n\big(\mathbb{R}Hom_{A \otimes A^{op}}(A, N)\big) \cong Ext^n_{A \otimes A^{op}}(A, N),$$

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[2]also see [CV, BGNX, C] for rigorous explicit construction of the underlying graded commutative multiplication, called the sphere product

where $\mathbb{R}Hom$ denotes the derived mapping space, while the Hochschild homology groups $HH_\bullet(A, N) \cong Tor_n^{A \otimes A^{op}}(A, N)$ are defined similarly by derived tensor products. These (co)homology groups are given by standard (co)chain complexes [Ge, L]. Hochschild cohomology of any (dg-)associative algebra has a natural Gerstenhaber algebra structure and further, by the (solutions to the) Deligne conjecture, the latter is induced by an E_2-algebra structure on the Hochschild cochains.

Hochschild (co)chains have been used as models for free loop spaces since at least the 1980s. Indeed, there is an isomorphism (see [CV, FTV], for example)

$$(1) \qquad H_\bullet(LM) \cong HH^\bullet(C^*(M), C_*(M)) \cong HH^\bullet(C^*(M), C^*(M))[d]$$

if M is an oriented and simply connected manifold of dimension d which, in characteristic zero is an isomorphism of Gerstenhaber algebras [FT]. Further, Hochschild chains of a Calabi-Yau algebra carries a topological conformal field theory structure [Lu4]. The above isomorphisms (1) make use of two ingredients. First, it uses the (dual of) an isomorphism $HH_\bullet(C^*(M), C_*(M)) \cong H_\bullet(LM)$ for any simply connected space M (which can be described in geometric terms by Chen interated integrals when M is a manifold) and, second, it uses a lift of the Poincaré duality quasi-isomorphism $C^*(M) \to C_*(M)[\dim(M)]$ to a bimodule map, when M is further a closed manifold.

In this paper, we generalize these two facts from circles to n-dimensional spheres as well as the E_2-algebra structure on Hochschild cochains as we explain below. Combining these three ingredients will give us the desired E_{n+1}-algebra structure on $C_*(Map(S^n, M))$. Our technique should be related to those of Hu [Hu] and Hu-Kriz-Voronov [HKV].

Bimodules over an associative algebra correspond to the *operadic* notion of E_1-modules. There is a notion of E_n-Hochschild cohomology where maps of A-bimodules are replaced by maps of A-E_n-modules for an E_n-algebra A ([L-HA, F1, Fre]). The Kontsevich-Soibelman generalization of the Deligne conjecture, *i.e.*, the higher Deligne conjecture, is that the E_n-Hochschild cohomology of A, denoted $HH_{\mathcal{E}_n}(A, A)$ is an E_{n+1}-algebra. For X a topological space, the cochains $C^*(X)$ are more than simply an associative algebra but are *homotopy commutative*, that is, it carries a functorial structure of an E_∞-algebra; in particular of an E_n-algebra for all n. In characteristic zero, one can use CDGAs models for the cochains, but this is not possible when working over the integers or a finite field. Nevertheless, for E_∞-algebras, E_n-Hochschild cohomology have extra functoriality (not shared by all E_n-algebras) and actually identifies with higher Hochschild cohomology over the n-spheres.

The latter theories are the subject of Section 3 and can be expressed in terms of *factorization homology*, also referred to as *topological chiral homology* [L-HA, F1, CG, GTZ2]. Factorization homology is an invariant of *both* (framed) manifolds (and framed embeddings) and E_n-algebras based on (extended) topological field theories. In fact, the factorization homology of E_∞-algebras becomes a homotopy invariant and can be applied to any space (and continuous maps) and not just to framed manifolds. This generalization is precisely computed by *higher Hochschild homology*, introduced by Pirashvili in [P], which can be seen as a kind of *limit* of these ideas when the dimension of the TFT goes to infinity [GTZ2]. Indeed, by Theorem 3.13 below (and [GTZ2, F1, L-HA]) if X is a manifold and A is an E_∞-algebra, then, the factorization homology $\int_X A$ of X with coefficients in A is naturally equivalent to the Hochschild chains $CH_X(A)$ of A over X.

The restriction to E_∞-algebras is not an issue in our case of interest since the cochain complex $C^*(X)$ is indeed an E_∞-algebra. We study the *higher Hochschild (co)chains* for E_∞-algebras and modules in Section 3.1, which are modeled over spaces in the same way the usual Hochschild (co)chains are modeled on circles. More precisely, this is a rule that assigns to any space X, E_∞-algebra A, and A-module M, a chain complex $CH_X(A, M)$, *functorial in every argument*, such that for $X = S^1$, one recovers the usual Hochschild chains. The functoriality with respect to spaces is a key feature which allows us to derive algebraic operations on the higher Hochschild (co)chain complexes from maps of topological spaces.

Higher Hochschild chains have a good axiomatic characterization (similar to Eilenberg-Steenrod axioms) which formally follows from the fact that E_∞-algebras are tensored over spaces, see Corollary 3.29 in Section 3.3. This allows to generalize the aforementioned relationship between free loop spaces and Hochschild chains to every space. In fact, we prove (Theorem 3.33) that there is a natural map of E_∞-algebras $CH_Y(C^*(X)) \to C^*(Map(Y, X))$ which is a quasi-isomorphism when X is $\dim(Y)$-connected. This is an E_∞-analogue of our previous result [GTZ] for CDGA's in characteristic zero (using generalizations of Chen *iterated integrals*).

In Section 4, we study the algebraic structure of higher Hochschild cochains. We first define, for any E_∞-A-algebra B, the (associative) *wedge product*

$$CH^X(A, B) \otimes CH^Y(A, B) \to CH^{X \vee Y}(A, B)$$

and then we prove that when X is a sphere S^d, the wedge product induces a structure of E_d-*algebra* on $CH^{S^d}(A, B)$, generalizing the usual cup-product in Hochschild cohomology.

Theorem 4.12. *Let A be an E_∞-algebra and B an E_∞-A-algebra. The collection of maps $(pinch_{S^d, k} : \mathcal{C}_d(k) \times S^d \longrightarrow \bigvee_{i=1\ldots k} S^d)_{k \geq 1}$ makes $CH^{S^d}(A, B)$ into an E_d-algebra, such that the underlying E_1-structure of $CH^{S^d}(A, B)$ agrees with the one given by the* cup-product,

$$\cup_{S^d} : \quad CH^{S^d}(A, B) \otimes CH^{S^d}(A, B) \longrightarrow CH^{S^d \vee S^d}(A, B) \longrightarrow CH^{S^d}(A, B).$$

The CDGA version of this result goes back to the first author's note [G1]. This result is in fact a relative version of the higher Deligne conjecture.

In Section 6, we reinterpret and generalize the above results, to the case of all E_n-algebras in terms of *centralizers* of E_n-algebra maps. The latter are E_n-algebras satisfying a universal property whose existence was established by Lurie [L-HA]. Their importance lies in the fact that their structure controls relative deformations of categories of E_n-modules. We prove

Theorem (Theorem 6.8 and Proposition 6.22). *Let $f : A \to B$ be an E_n-algebra map. The $\mathcal{E}_n$-Hochschild cohomology $HH_{\mathcal{E}_n}(A, B) \cong \mathbb{R}Hom_A^{\mathcal{E}_n}(A, B)$ has a natural E_n-algebra structure exhibiting it as the centralizer $\mathfrak{z}(f)$ of f.*

Our result gives another proof of existence of centralizers and also gives an *explicit description* in terms of factorization algebras. Applying the universal property of centralizers when $f = id_A$, and using an approach due to Lurie [L-HA], we obtain that $\mathfrak{z}(id_A) \cong HH_{\mathcal{E}_n}(A, A)$ inherits a canonical E_{n+1}-algebra structure, giving a *solution to the higher Deligne conjecture*, see Corollary 6.28. We also prove that the Hochschild cochains $CH^{S^d}(A, A)$ of a commutative algebra A are equivalent to its E_n-Hochschild cohomology (Proposition 6.33).

As already mentioned, our approach is based on the relationship between E_n-algebras and factorization algebras which we briefly explain, among other preliminaries, in Section 2.

Factorization algebras originated from quantum field theories and the pioneering work of Beilinson-Drinfeld [BD] on chiral and vertex algebras. We follow an approach due to Lurie [L-HA] and Costello-Gwilliam [CG]. They are algebraic structures which share many similarities with (co)sheaves and were introduced to describe quantum field theories, much in the same way the sheaf of functions describes the structure of a manifold or scheme [BD, CG]. Roughly speaking a factorization algebra $\mathcal{F}$ associate (covariantly) cochain complexes to open subsets of a (stratified) manifold X together with multiplications $\mathcal{F}(U_1) \otimes \cdots \otimes \mathcal{F}(U_n) \to \mathcal{F}(V)$ for any family of pairwise disjoint open subsets of an open set V in X. It is required to satisfy a "cosheaf-like" condition, meaning that $\mathcal{F}(V)$ can be computed by analogues of Čech complexes indexed on nice enough covers.

Factorization homology is a catchword to describe homology theories *specific* to, say, oriented[3] manifolds of a fixed dimension n. It can be seen as the (derived) global section of (locally constant) factorization algebras much in the same way as singular cohomology can be seen as sheaf cohomology with value in a constant sheaf.

E_n-algebras can be identified with factorization algebras on $\mathbb{R}^n$ which are locally constant, that is for which the structure map $\mathcal{F}(U) \to \mathcal{F}(V)$ is an equivalence when U is a subset of V and both are homeomorphic to a disk. This provides a nice model for the category of E_n-algebras, which, in some sense can be thougt as a kind of mild "strictification" of E_n-algebras. This is the model we use in our approach to centralizers. In Section 5.2, we recall the relationship between E_n-modules over an E_n-algebra A and factorization homology over $S^{n-1} \times \mathbb{R}$. Namely that the category of E_n-A-modules is equivalent to the category of left modules over the (associative) algebra $\int_{S^{n-1} \times \mathbb{R}} A$. For $n = \infty$, one recovers that E_∞-A-modules are the same as left modules over A ([L-HA, Lu2, KM]).

Theorem 5.13. *Let A be an E_∞-algebra. There is an equivalence of symmetric monoidal ∞-categories between the category A-Mod^{E_∞} of E_∞ A-Modules and the category of left A-modules (where A is viewed as an E_1-algebra).*

We give a proof of this result using factorization homology in Section 5.3. From this, we deduce in Section 5.4, that, the Poincaré duality isomorphism can be *uniquely lifted* into an E_∞-quasi-isomorphism

Corollary 5.26. *Let $(X, [X])$ be a Poincaré duality space. The cap-product by $[X]$ induces a quasi-isomorphism of E_∞-$C^*(X)$-modules*

$$C^*(X) \xrightarrow{\simeq} C_*(X)[\dim(X)]$$

realizing the (unique) E_∞-lift of the Poincaré duality isomorphism.

Putting together the above results on the Deligne conjecture, Poincaré duality and interpretation of higher Hochschild chains in terms of mapping spaces, we obtain in Section 7 that the chains $C_*(Map(S^n, M))$ for an n-connected manifold M inherits a natural E_{n+1}-algebra structure (Theorem 7.1) lifting Sullivan-Voronov

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[3]there are also variants specific to many other classes of structured manifold of fixed dimension; for instance framed, spin or unoriented ones

sphere product. To identify the sphere product, we use the fact that we have an *explicit* description of the centralizers of E_n-algebra maps. The above results yield chain level constructions over any field or the ring of integers. Results similar to Theorem 7.1 can be obtained using only bimodules maps (not necessarily quasi-isomorphisms) $C^*(M) \to C_*(M)[d]$. This yield a *functorial* construction of E_{n+1}-algebra structures on $C_*(Map(S^n, M))$, see Theorem 7.10.

Furthermore, ideas similar (in some sense dual) to the concept of the centralizer for E_n-algebra maps, lead to a description and construction of bar and iterated bar constructions for E_n-algebras. The bar construction of a (dg-)associative *augmented* algebra is a standard functor in homological algebra and algebraic topology. The bar construction of a CDGA is itself an augmented algebra and thus can be iterated; this result was extended to E_∞-algebras by Fresse [Fre2].

In topology, *iterated bar constructions* arise as models for *iterated loop spaces* $\Omega^n(X)$, the space of pointed maps from the sphere S^n to a pointed space X. The latter is an E_n-algebra in the category of spaces so that its singular cochains becomes an E_n-coalgebra in E_∞-algebras. In Section 8.1, for an augmented E_∞-algebra A we apply the E_n-algebra structure on higher Hochschild chains $CH^{S^n}(A, k)$ (identified with the centralizer construction for the augmentation $A \to k$) to describe the *iterated Bar construction* of an augmented E_∞-algebra. We obtain that $Bar(A)$ is naturally an E_1-coalgebra in augmented E_∞-algebras so that we can iterate the construction. With this, we prove that *the n-iterated bar construction $Bar^{(n)}$ is an E_n-coalgebra inside the $((\infty, 1))$-category of E_∞-algebras*, see Theorem 8.9. We then relate this construction to iterated loop spaces by showing that there is a natural map of E_n-coalgebras (and E_∞-algebras) $Bar^{(n)}(C^*(X)) \to C^*(\Omega^n(X))$ which is a quasi-isomorphism if X is n-connected.

Corollary 8.10. *Let Y be a topological space. The map $\mathcal{I}t^{\Omega^n} : Bar^{(n)}(C^*(Y)) \to C^*(\Omega^n(Y))$ is an E_n-coalgebra morphism in the category of E_∞-algebras, which is an equivalence if Y is n-connected.*

We also give similar dual statements for *chains on iterated loop spaces* using that the dual of the Bar construction is precisely the centralizer $\mathfrak{z}(A \to k) \cong CH^{S^n}(A, k)$ of the augmentation $A \to k$.

In Section 8.2, we consider the *bar construction of an E_m-algebra A*. Using its factorization homology interpretation due to Francis [F1], we prove that the bar construction $Bar^{(1)}(A)$ is naturally an E_{m-1}-algebra which allows us to iterate this construction up to m-times. Then, using the technique of Section 6, we prove that

Theorem 8.37. *The n-iterated bar construction of an augmented E_m-algebra ($m \geq 1$) has a natural structure of an E_n-coalgebra inside the $((\infty, 1)-)$category E_{m-n}-algebras.*

Similar result are stated in [F2]. The existence of the iterated bar construction for E_m-algebras as a chain complex was proved in [Fre3]. The idea behind the theorem is again to prove a similar statement for locally constant factorization algebras over $\mathbb{R}^m$. More precisely, we prove that an augmented locally constant factorization algebra over $\mathbb{R}^m$ naturally gives rise to a locally constant *stratified* factorization algebra on the pointed sphere S^m whose global sections are precisely the iterated bar construction $Bar^{(m)}(A)$. This construction extends into a locally constant factorization *coalgebra* over $\mathbb{R}^m$ which associates to any disk (the global

sections of) a stratified factorization algebra on the one-point compactification of
the disk.

In this paper, we work in Lurie's framework of stable ∞-categories [L-HTT,
L-HA], which is very well suited for doing homological algebra in the symmetric
monoidal context. In particular, we will work over the (derived) $(\infty, 1)$-category
$k\text{-Mod}_\infty$ of chain complexes over a commutative unital ring k. (In section 2, we
briefly recall notions of $(\infty, 1)$-categories, ∞-operads and in particular the E_n-
operad and its algebras and their modules.) It should be noted that in characteristic
zero, one can use CDGA's instead of E_∞-algebras which allows us to have (model)
categories interpretation of all our results in the spirit of [G1, GTZ, GTZ2].

Conventions and notations:

(1) We use *homological grading*, emphasizing the geometric dimension of the
    chains on mapping spaces. In particular, unless otherwise stated, differen-
    tial will lower the degree by one. We will write $k\text{-Mod}_\infty$ for the $(\infty, 1)$-
    category of chain complexes of k-modules and $\otimes$ for tensor products over
    the ground ring k.

(2) We will denote the *Hochschild chain complex* of A, modeled over a space,
    X with values in an A-module M, by $CH_X(A, M)$ as an object in the
    stable $(\infty, 1)$-category of chain complexes. This is a *covariant* functor in
    X. Similarly, we will also denote the *Hochschild cochain complex* of A,
    modeled over a space X, with values in an A-module M, by $CH^X(A, M)$,
    as an object in the stable $(\infty, 1)$-category of chain complexes. This is a
    contravariant functor of X, see § 3.2. This is compatible with the notation
    introduced in [GTZ2] but not with those in [G1, GTZ]. We choose this
    notation in order to emphasize the variance of the functor with respect to
    X.

(3) We will respectively denote $HH^{X,n}(A, M)$ and $HH_{X,n}(A, M)$ the degree n
    homology groups of $CH^X(A, M)$ and $CH_X(A, M)$.

(4) For $n \in \mathbb{N} \cup \{\infty\}$, we will write $E_n\text{-Alg}$ for the $(\infty, 1)$-category of E_n-
    algebras in $k\text{-Mod}_\infty$ as studied in [Lu3, L-HA, F1]. We will also denote
    by $\mathbb{E}_n^\otimes$ the ∞-operad governing E_n-algebras, $HH_{\mathcal{E}_n}(A, M)$ for the E_n-
    Hochschild cohomology of an E_n-algebra with value in an E_n-A-module
    M (Definition 6.1) and $\int_X A$ for the factorization homology of A on a
    framed manifold X (see § 2.3). Also $CDGA_\infty$ will be the $(\infty, 1)$-category
    of commutative differential graded k-algebras (CDGA for short).

(5) Given an E_n-algebra A, we will write $A\text{-}Mod^{E_n}$ for the $(\infty, 1)$-category of
    E_n-modules over A. Similarly, if B is an E_m-algebra (with $m \geq n$), we will

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write $B\text{-}Mod^{E_n}$ for the $(\infty, 1)$-category of E_n-modules over B viewed as an E_n-algebra.

(6) If A is an E_n-algebra $(n \geq 1)$ by a left or right module over A, we mean a left or right module over A viewed as an E_1-algebra. We will denote $A - LMod$ and $A - RMod$ the respective $(\infty, 1)$-categories of left and right modules over A.

(7) If A is a CDGA, and V an A-module, we will write $\mathrm{Sym}_A(V)$ for its differential graded symmetric algebra. When $A = k$, we will often simply write $S(V) := \mathrm{Sym}_k(V)$.

(8) Unless otherwise stated, we will work in the context of unital algebras.

2. Preliminaries on E_n-algebras and factorization homology

In this section, we briefly recall notions of $(\infty, 1)$-categories, ∞-operads and in particular the E_n-operad, its algebras, and their modules. There are several equivalent notions of (symmetric monoidal) $(\infty, 1)$-categories and the reader should feel free to use its favorite ones. Below, we recall very briefly the model given by the complete Segal spaces and give some examples.

2.1. ∞-**categories.** Following [R, Lu4], an $(\infty, 1)$-category is a *complete Segal space*. There is a simplicial closed model category structure, denoted $\mathcal{SeSp}$ on the category of simplicial spaces such that a fibrant object in the $\mathcal{SeSp}$ is precisely a Segal space. Rezk has shown that the category of simplicial spaces has another simplicial closed model structure, denoted $\mathcal{CSeSp}$, whose fibrant objects are precisely complete Segal spaces [R, Theorem 7.2]. Let $\mathbb{R} : \mathcal{SeSp} \to \mathcal{SeSp}$ be a fibrant replacement functor. Let $\widehat{\cdot} : \mathcal{SeSp} \to \mathcal{CSeSp}$, $X_\bullet \to \widehat{X_\bullet}$, be the completion functor that assigns to a Segal space an equivalent complete Segal space. The composition $X_\bullet \mapsto \widehat{\mathbb{R}(X_\bullet)}$ gives a fibrant replacement functor $L_{\mathcal{CSeSp}}$ from simplicial spaces to complete Segal spaces.

Let us explain how to go from a model category to a simplicial space. The standard key idea is to use Dwyer-Kan localization. Let $\mathcal{M}$ be a model category and $\mathcal{W}$ be its subcategory of weak-equivalences. We denote $L^H(\mathcal{M}, \mathcal{W})$ its *hammock localization*, see [DK]. One of the main properties of $L^H(\mathcal{M}, \mathcal{W})$ is that it is a simplicial category and that the (usual) category $\pi_0(L^H(\mathcal{M}, \mathcal{W}))$ is the homotopy category of $\mathcal{M}$. Furthermore, every weak equivalence has a (weak) inverse in $L^H(\mathcal{M}, \mathcal{W})$. When $\mathcal{M}$ is further a simplicial model category, then for every pair (x, y) of objects $Hom_{L^H(\mathcal{M}, \mathcal{W})}(x, y)$ is naturally homotopy equivalent to the derived mapping space $\mathbb{R}Hom(x, y)$.

It follows that any model category $\mathcal{M}$ gives rise functorially to the simplicial category $L^H(\mathcal{M}, \mathcal{W})$. Taking the nerve $N_\bullet(L^H(\mathcal{M}, \mathcal{W}))$ we obtain a simplicial space. Composing with the complete Segal Space replacement functor we get a functor $\mathcal{M} \to L_\infty(\mathcal{M}) := L_{\mathcal{CSeSp}}(N_\bullet(L^H(\mathcal{M}, \mathcal{W})))$ from model categories to $(\infty, 1)$-categories (that is complete Segal spaces).

Example 2.1. Applying the above procedure to the model category of simplicial sets $sSet$, we obtain the $(\infty, 1)$-category $sSet_\infty$. Similarly from the model category $CDGA$ of commutative differential graded algebras, which we refered to as CDGAs for short, we obtain the $(\infty, 1)$-category $CDGA_\infty$. Note that a simplicial space is determined by its $(\infty, 0)$ path groupoid and therefore the category of simplicial sets should be thought of as the $(\infty, 1)$ category of all $(\infty, 0)$ groupoids. Further,

the disjoint union of simplicial sets and the tensor products (over k) of algebras are monoidal functors which gives $sSet$ and $CDGA$ a structure of monoidal model category (see [Ho] for example). Thus $sSet_\infty$ and $CDGA_\infty$ also inherit the structure of symmetric monoidal $(\infty, 1)$-categories in the sense of [R, Lu4].

The model category of topological spaces yields the $(\infty, 1)$-category Top_∞. Since $sSet$ and Top are Quillen equivalent [GJ, Ho], their associated $(\infty, 1)$-categories are equivalent (as $(\infty, 1)$-categories): $sSet_\infty \overset{\sim}{\underset{\sim}{\rightleftarrows}} Top_\infty$, where the left and right equivalences are respectively induced by the singular set and geometric realization functors.

One can also consider the pointed versions $sSet_{\infty*}$ and $Top_{\infty*}$ of the above $(\infty, 1)$-categories (using the model categories of these pointed versions [Ho]).

Example 2.2. There are model categories A-Mod and A-$CDGA$ of modules and commutative algebras over a CDGA A, thus the above procedure gives us $(\infty, 1)$-categories A-Mod_∞ and A-$CDGA_\infty$ and the base changed functor lifts to an $(\infty, 1)$-functor. Further, if $f : A \to B$ is a weak equivalence, the natural functor $f_* : B$-$Mod \to A$-Mod induces an equivalence B-$Mod_\infty \overset{\sim}{\to} A$-$Mod_\infty$ of $(\infty, 1)$-categories since it is a Quillen equivalence.

Moreover, if $f : A \to B$ is a morphism of CDGAs, it induces a natural functor $f^* : A$-$Mod \to B$-$Mod, M \mapsto M \otimes_A B$, which is an equivalence of $(\infty, 1)$-categories when f is a quasi-isomorphism, and is a (weak) inverse of f_* (see [TV] or [KM]). Here, we also denote $f^* : A$-$Mod_\infty \to B$-Mod_∞ and $f_* : B$-$Mod_\infty \to A$-Mod_∞ the (derived) functors of $(\infty, 1)$-categories induced by f. Since we are working over a field of characteristic zero, the same results applies to monoids in A-Mod and B-Mod, that is to the categories A-$CDGA_\infty$ and B-$CDGA_\infty$.

Also, note that if A, B, C are CDGAs and $f : A \to B$, $g : A \to C$ are CDGAs morphisms, we can form the (homotopy) pushout $D \cong B \otimes_A^{\mathbb{L}} C$; let us denote $p : B \to D$ and $q : C \to D$ the natural CDGAs maps. Then, we get the two natural based change $(\infty, 1)$-functors C-$Mod_\infty \overset{f^* \circ g_*}{\underset{p_* \circ q^*}{\rightrightarrows}} B$-$Mod_\infty$. Given any $M \in C$-Mod, the natural map $f^* \circ g_*(M) \to p_* \circ q^*(M)$ is an equivalence [TV, Proposition 1.1.0.8].

2.2. ∞-operads, E_n-algebras. An operad is a special case of a colored operad which itself is a special case of an ∞-operad. An *infinity operad* $\mathcal{O}$ is an ∞-category together with a functor $\mathcal{O}^\otimes \to N(Fin_*)$ satisfying a list of axioms, see [L-HA]. An other (equivalent) approach to ∞-operads is given by the dendroidal sets [CiMo].

The simplest example of an ∞-operad is $N(Fin_*) \to N(Fin_*)$. This example is the ∞-operad associated to the operad $Comm$. In other words $Comm^\otimes = N(Fin_*)$.

Definition 2.3. The configuration spaces of small n-dimensional cubes embedded in a bigger n-cube form an operad, $\mathbb{E}_n$, whose associated ∞-operad is denoted by $\mathbb{E}_n^\otimes$ see [L-HA]. This example has the same objects as Fin_*, and we will denote $\mathbb{E}_n^\otimes(I, J)$ its spaces of morphisms from I to J.

There is a standard model for this operad given by $\big(\mathcal{C}_n(r)\big)_{r \geq 1}$, the *operad of little n-cubes*, see [Ma, L-HA], where $\mathcal{C}_n(r)$ is the configuration space of rectilinear embeddings of r-disjoint cubes inside an unit cube. Its singular chain $C_*(\mathcal{C}_n(r))$ is a model for the operad governing E_n-algebras in k-Mod_∞.

Recall that, for any integer $n \geq 0$, $\mathbb{E}_n^{\otimes}$ denotes the ∞-operad of little n-cubes. By an $\mathbb{E}_n$-algebra we mean an algebra over the ∞-operad $\mathbb{E}_n^{\otimes}$. We will denote E_n-Alg the symmetric monoidal ∞-category of $\mathbb{E}_k$-algebras in the symmetric monoidal ∞-category of differential graded k-modules (which is equivalent to the one given in Definition 2.4 below).

For any $\mathbb{E}_n$-algebra A, let $A\text{-}Mod^{E_n}$ denote the symmetric monoidal ∞-category of modules over the $\mathbb{E}_n$-algebra A. If $\mathcal{C}$ is a symmetric monoidal $(\infty, 1)$-category (different from $k\text{-Mod}_\infty$), *we denote E_n-Alg$(\mathcal{C})$ for the $(\infty,1)$-category of E_n-algebras in $\mathcal{C}$* and similarly E_n-coAlg$(\mathcal{C})$ for the category of E_n-coalgebras in $\mathcal{C}$.

There are natural maps (sometimes called the stabilization functors)

$$(2) \qquad\qquad \mathbb{E}_0^{\otimes} \longrightarrow \mathbb{E}_1^{\otimes} \longrightarrow \mathbb{E}_2^{\otimes} \longrightarrow \cdots$$

(induced by taking products of cubes with the interval $(-1, 1)$). It is a fact that the colimit of this diagram, denoted by $\mathbb{E}_\infty^{\otimes}$ can be identified with $Comm^{\otimes}$ [L-HA, Section 5.1]. In particular, for any $n \in \mathbb{N} - \{0\} \cup \{+\infty\}$, the map $\mathbb{E}_1^{\otimes} \to \mathbb{E}_n^{\otimes}$ induces a functor E_n-Alg $\to E_1$-Alg which associates to an E_n-algebra its underlying E_1-algebra structure.

According to Lurie [L-HA] (also see [F1, AFT]), we also have an alternative definition for E_n-algebras.

Definition 2.4. The $(\infty, 1)$-category of E_n-*algebras*, is defined as the $((\infty, 1)$-)category of symmetric monoidal functors

$$Fun^{\otimes}(Disk_n^{fr}, k\text{-Mod}_\infty)$$

where $Disk_n^{fr}$ is the category with objects the integers and morphism the spaces $Disk_n^{fr}(k, \ell) := Emb^{fr}(\coprod_k \mathbb{R}^n, \coprod_\ell \mathbb{R}^n)$ of framed embeddings of k disjoint copies of a disk $\mathbb{R}^n$ into ℓ such copies; the monoidal structure is induced by disjoint union of copies of disks.

We will denote by $Map_{E_n\text{-Alg}}(A, B)$ the mapping space of E_n-algebras maps from A to B, *i.e.*, the space of maps between the associated symmetric monoidal functors.

In other words, E_n-Alg is equivalent to the $(\infty, 1)$-category of $Disk_n^{fr}$-algebras (where $Disk_n^{fr}$ is equipped with its obvious ∞-operad structure). The tensor products in $k\text{-Mod}_\infty$ induces a symmetric monoidal structure on E_n-Alg as well (which, for instance, can be represented by usual Hopf operads such as those arising from the filtration of the Barratt-Eccles operad [BF]).

Example 2.5 (Opposite of an E_n-algebra)**.** There is a canonical $\mathbb{Z}/2\mathbb{Z}$-action on E_n-Alg induced by the antipodal map $\tau : \mathbb{R}^n \to \mathbb{R}^n$, $x \mapsto -x$ acting on the source of $Fun^{\otimes}(Disk_n^{fr}, k\text{-Mod}_\infty)$. If A is an E_n-algebra, then the result of this action $A^{op} := \tau^*(A)$ is its opposite algebra. If $n = \infty$, the antipodal map is homotopical to the identity so that A^{op} is equivalent to A as an E_∞-algebra.

Example 2.6 (Singular (co)chains)**.** Let X be a topological space. Then its *singular cochain complex $C^*(X)$ has a natural structure of an E_∞-algebra*, whose underlying E_1-structure is given by the usual (strictly associative) cup-product (for instance see [M2]). Similarly, the *singular chains $C_*(X)$ have a natural structure of E_∞-coalgebra* which is the predual of $(C^*(X), \cup)$. There are similar explicit constructions for simplicial sets $X_\bullet$ instead of spaces, see [BF].

We recall that $C^*(X)$ is the linear dual of the singular chain complex $C_*(X)$ which is the geometric realization (in k-Mod$_\infty$) of the simplicial k-module $k[\Delta_\bullet(X)]$ spanned by the singular set $\Delta_\bullet(X) := Map(\Delta^\bullet, X)$. Here, Δ^n is the standard n-dimensional simplex.

Also note that, for E_∞-algebras A, B, the mapping space $Map_{E_\infty\text{-Alg}}(A, B)$ is the (geometric realization of the) simplicial set $[n] \mapsto Hom_{E_\infty\text{-Alg}}\big(A, B \otimes C^*(\Delta^n)\big)$.

Remark 2.7. The $(\infty, 1)$-category E_∞-Alg is enriched over $sSet_\infty \cong Top_\infty$ and has all $(\infty\text{-})$colimits. In particular, it is *tensored* over $sSet_\infty$, see [L-HTT, L-HA] for details on tensored ∞-categories (and [Ke] for the classical theory) or, for instance, [EKMM, MCSV] in the E_∞-case (in the context of topologically enriched model categories). We recall that it means that there is a functor E_∞-Alg $\times$ $sSet_\infty \to E_\infty$-Alg, denoted $(A, X_\bullet) \mapsto A \boxtimes X_\bullet$, together with natural equivalences

$$Map_{E_\infty\text{-Alg}}\big(A \boxtimes X_\bullet, B\big) \cong Map_{sSet_\infty}\big(X_\bullet, Map_{E_\infty\text{-Alg}}(A, B)\big).$$

Note that the tensor $A \boxtimes X_\bullet$ can be computed as the colimit $\varinjlim p^A_{X_\bullet}$ where $p^A_{X_\bullet}$ is the constant map $X_\bullet \to E_\infty$-Alg taking value A, for instance see [L-HTT, Corollary 4.4.4.9]. Similarly, $CDGA_\infty$ is tensored over $sSet_\infty$ (and thus Top_∞ as well).

We will use the following fact, which identifies the coproduct in E_∞-Alg with the tensor product, to show the Hochschild complex of an E_∞-algebra model over a space X has a natural E_∞ structure.

Proposition 2.8. *In the symmetric monoidal $(\infty, 1)$-category E_∞-Alg, the tensor product is a coproduct.*

For a proof see Proposition 3.2.4.7 of [L-HA] (or [KM, Part V, Corollary 3.4]); this essentially follows from the observation that an E_∞-algebra is a commutative monoid in $(k$-Mod$_\infty, \otimes)$, see [L-HA] or [KM, Section 5.3]. In particular, Proposition 2.8 implies that, for any finite set I, $A^{\otimes I}$ has a natural structure of E_∞-algebras which can be rephrased as

Proposition 2.9. *A symmetric monoidal functor $N(Fin) \to k$-Mod$_\infty$ has a natural lift to an ∞-functor $N(Fin) \to E_\infty$-Alg.*

It follows that, for a finite set I, we have natural multiplication maps

$$A^{\otimes I} \xrightarrow{m_A^{(I)}} A$$

which is a map in E_∞-Alg and is compatible with compositions. We will simply write $m_A : A \otimes A \to A$ for the E_∞-algebra map obtained by taking $I = \{0, 1\}$. Any A-module can be pulled back along $m_A^{(I)}$ to inherit a canonical $A^{\otimes I}$-module structure:

Proposition 2.10. *The maps $m_A^{(I)}$ induced by the functor of Proposition 2.9 yields a natural (in A) $(\infty\text{-})$functors A-Mod$^{E_\infty} \xrightarrow{(m_A^{(I)})^*} A^{\otimes I}$-Mod$^{E_\infty}$ lifting the usual base-change functor for commutative algebras.*

Let Fin and Fin_* denote the categories of finite sets and pointed finite sets respectively. There is a forgetful functor $Fin_* \to Fin$ forgetting which point is the base point. There is also a functor $Fin \to Fin_*$ which adds an extra point called the base point.

Further, since the ∞-operads $\mathbb{E}_n^{\otimes}$ are coherent (see [L-HA, Lu3]), the *categories*
A-Mod^{E_n} for $A \in E_n$-Alg assembles to form an $(\infty, 1)$-category of all E_n-algebras
and their modules, denoted Mod^{E_n} (or $Mod^{E_n}(\mathcal{C})$ when we want to emphasize $\mathcal{C}$).
The canonical functor $Fin \to Fin_*$ adding a base point yields a canonical functor [5]
$\iota : Mod^{E_n}(\mathcal{C}) \to Alg_{E_n}(\mathcal{C})$ which gives rise, for any E_n-algebra A, to a (homotopy)
pullback square:

$$(3) \qquad \begin{array}{ccc} A\text{-}Mod^{E_n} & \longrightarrow & Mod^{E_n} \\ \downarrow & & \downarrow{\scriptstyle \iota} \\ \{A\} & \longrightarrow & E_n\text{-Alg} \end{array}$$

We refer to [Lu2, L-HA, F1] for details. Note that the functor ι is monoidal.
Further, if $A \xrightarrow{f} B$, $A \xrightarrow{g} C$ are two maps of E_∞-algebras, and $M \in B\text{-}Mod^{E_\infty}$ and
$N \in C\text{-}Mod^{E_\infty}$, then

$$(4) \qquad \iota\left(M \overset{\mathbb{L}}{\underset{A}{\otimes}} N\right) \quad \cong \quad B \overset{\mathbb{L}}{\underset{A}{\otimes}} C.$$

Example 2.11. If $n = 1$, $A\text{-}Mod^{E_1}$ is equivalent to the $(\infty, 1)$-category of A-
bimodules and if $n = \infty$, $A\text{-}Mod^{E_\infty}$ is equivalent to the $(\infty, 1)$-category of left A-
modules, see [Lu2, L-HA] (and Proposition 5.8, Theorem 5.13 below). In general,
$A\text{-}Mod^{E_n}$ can be described in terms of factorization homology of A, see § 5.2.

2.3. **Factorization algebras and factorization homology.**

Definition 2.12. Given a topological manifold M of dimension n, one can define a
colored operad whose objects are open subsets of M that are homeomorphic to $\mathbb{R}^n$
and whose morphisms from $\{U_1, \cdots, U_n\}$ to V are empty except when the U_i's are
mutually disjoint subsets of V, in which case they are singletons. The ∞-operad
associated to this colored operad is denoted by $N(\text{Disk}(M))$, see [L-HA], Remark
5.2.4.7.

Unfolding the definition we find that an $N(\text{Disk}(M))$-algebra on a manifold
M, with value in chain complexes, is a rule that assigns to any open disk[6] U a
chain complex $\mathcal{F}(U)$ and, to any finite family of disjoint open disks $U_1, \ldots, U_n \subset V$
included in a disk V, a natural map $\mathcal{F}(U_1) \otimes \cdots \otimes \mathcal{F}(U_n) \to \mathcal{F}(V)$. An $N(\text{Disk}(M))$-
algebra is **locally constant** if for any inclusion of open disks $U \hookrightarrow V$ in X, the
structure map $\mathcal{F}(U) \to \mathcal{F}(V)$ is a quasi-isomorphism (see [L-HA, Lu3]) .

Locally constant $N(\text{Disk}(M))$-algebra are actually (homotopy) *locally constant
factorization algebras* in the sense of Costello [CG, C], see Remark 2.17 below.

A *locally constant factorization coalgebra* is an $N(\text{Disk}(M))$-coalgebra such that
for any inclusion of open disks $U \hookrightarrow V$ in X, the structure map $\mathcal{F}(V) \to \mathcal{F}(U)$ is
a quasi-isomorphism.

Notation: we denote Fac_M^{lc} the *$(\infty, 1)$-category of locally constant $N(Disk(M))$-
algebras* (see [CG, G2]). We will also denote $N(\text{Disk}(M))$-Alg the $(\infty, 1)$-category
of $N(\text{Disk}(M))$-algebras (that is of prefactorization algebras).

If $\mathcal{A}$ is a locally constant $N(\text{Disk}(M))$-algebra, the rule which to an open disk
D associates the chain complex $\mathcal{A}(D)$ can be extended to any open set of M. In

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[5]which essentially forget the underlying module

[6]i.e. an open subset of M homeomorphic to a Euclidean ball

fact, Lurie has proved [L-HA, Lu3] that the functor $\mathrm{Disk}(M) \xrightarrow{\mathcal{A}} k\text{-Mod}_\infty$ has a left Kan extension along the embedding $\mathrm{Disk}(M) \hookrightarrow Op(M)$ where $Op(M)$ is the standard $((\infty, 1)\text{-})$category of open subsets of M, *i.e.*, with objects the open subsets of M and morphism from U to V are empty unless when $U \subset V$ in which case they are singletons.

Definition 2.13. Let M be a topological manifold and $\mathcal{A}$ be a locally constant factorization algebra.

Factorization homology is the $(\infty, 1)$-functor $Op(M) \otimes \mathrm{Fac}_M^{lc} \to k\text{-Mod}_\infty$, denoted $(M, \mathcal{A}) \mapsto \int_M \mathcal{A}$, given by the left Kan extension[7] of $\mathrm{Disk}(M) \xrightarrow{\mathcal{A}} k\text{-Mod}_\infty$. We say that $\int_M \mathcal{A}$ is the factorization homology of M with values in $\mathcal{A}$.

Remark 2.14. Factorization homology is also called topological chiral homology in [L-HA, Lu4, Lu3]. We prefer Francis terminology [F1, AFT, F2] which is justified by the fact that factorization homology is actually the homology (or said otherwise derived sections) of factorizations algebras in the sense of Costello [CG] as we proved [GTZ2], see Remark 2.17 below.

Example 2.15 (Framed manifolds). Let M be a framed manifold, then any E_n-algebra determines a locally constant factorization algebra on M (for instance, see [L-HA, F1, GTZ2, G2] or Theorem 2.20) so that one can define the factorization homology $\int_M A$. These locally constant factorization algebras are essentially constant, in the sense that they satisfy the property that there is a (globally defined) E_n-algebra A together with natural (with respect to the structure map of the factorization algebra) quasi-isomorphism $\mathcal{A}(D) \xrightarrow{\cong} A$ for every disk D. Thus, we call such a factorization algebra the *constant factorization algebra* on M associated to A. For instance, for $n = 0, 1, 3, 7$, there is a faithful embedding of E_n-algebras into locally constant factorization algebras over the n-sphere S^n.

For $M = \mathbb{R}^n$, one actually gets an equivalence between all locally constant factorization algebras over $M = \mathbb{R}^n$ and E_n-algebras, see Theorem 2.29 below.

Example 2.16. The canonical map $N(\mathrm{Disk}(M)) \to N(Fin_*)$ shows that any E_∞-algebras has a canonical structure of prefactorization algebra on any topological manifold M which turns out to actually be a locally constant factorization algebra. This construction is studied in detail (using the Hochschild chain models) in [GTZ2] and actually extends to define a factorization algebra on any CW-complex, *a fortiori* to any manifold with corners.

Remark 2.17. Let us justify a bit more the terminology of locally constant factorization algebras we are using (hoping it will avoid any possible confusion, also see [G2, § 4.2]). The notion of locally constant $N(\mathrm{Disk}(X))$-algebra is actually equivalent to the "full" notion of a locally constant factorization algebra on X in the sense of Costello [CG, C] which are a similar construction where the U_i are allowed to be any open subsets, satisfying a kind of "Čech/cosheaf-like" condition (and still being locally constant in the above sense). Let us now be more precise.

Definition 2.18. A **prefactorization** algebra is an algebra over the colored operad whose objects are open subsets of X and whose morphisms from $\{U_1, \cdots, U_n\}$ to V are empty unless when U_i's are mutually disjoint subsets of U, in which case they are singletons.

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[7]the existence of this extension is a non-trivial Theorem of [L-HA]

The above definition makes sense for *any* topological space X. Unfolding the definition, we find that a prefactorization algebra on X, with value in chain complexes, is a rule that assigns to any open set U a chain complex $\mathcal{F}(U)$ and, to any finite family of pairwise disjoint open sets $U_1, \ldots, U_n \subset V$ included in an open V, a natural map $\mathcal{F}(U_1) \otimes \cdots \otimes \mathcal{F}(U_n) \to \mathcal{F}(V)$. These structure maps are required to satisfy obvious associativity and symmetry conditions, see [CG]. They allow us to define "Čech-complexes" associated to a cover $\mathcal{U}$ of U. Denoting $P\mathcal{U}$ the set of finite pairwise disjoint open subsets $\{U_1, \ldots, U_n, U_i \in \mathcal{U}\}$, it is, by definition the chain (bi-)complex

$$\check{C}(\mathcal{U}, \mathcal{F}) = \bigoplus_{P\mathcal{U}} \mathcal{F}(U_1) \otimes \cdots \otimes \mathcal{F}(U_n) \leftarrow \bigoplus_{P\mathcal{U} \times P\mathcal{U}} \mathcal{F}(U_1 \cap V_1) \otimes \cdots \otimes \mathcal{F}(U_n \cap V_m) \leftarrow \cdots$$

where the horizontal arrows are induced by the alternating sum of the natural inclusions as for the usual Čech complex of a cosheaf (see [CG] or [GTZ2, G2]). The prefactorization algebra structure also induce a canonical map $\check{C}(\mathcal{U}, \mathcal{F}) \to \mathcal{F}(U)$.

Definition 2.19.   • A prefactorization algebra $\mathcal{F}$ on X is said to be a **factorization algebra** (in the sense of [CG]) if, for all open subsets $U \in Op(X)$ and for every factorizing cover[8] $\mathcal{U}$ of U, the canonical map $\check{C}(\mathcal{U}, \mathcal{F}) \to \mathcal{F}(U)$ is a quasi-isomorphism (see [C, CG]). Again, a factorization algebra is **locally constant** if for any inclusion of open disks $U \hookrightarrow V$ in X, the structure map $\mathcal{F}(U) \to \mathcal{F}(V)$ is a quasi-isomorphism.
   • In the above Definition, one can replace the symmetric monoidal ∞-category of chain complexes by any symmetric monoidal ∞-category $(\mathcal{C}, \otimes)$ which has all colimits, sifted limits and such that geometric realization (see [L-HTT]) distributes with respect to the monoidal structure, see [CG, G2].
   • A (resp. locally constant) *factorization coalgebra* on X with value in $(\mathcal{C}, \otimes)$ is a (resp. locally constant) factorization algebra on X with value in the opposite category $(\mathcal{C}^{op}, \otimes)$, that is an object of $\mathrm{Fac}_X(\mathcal{C}^{op})$ (resp. $\mathrm{Fac}_X^{lc}(\mathcal{C}^{op})$).

Notation: we denote $Fac_X(\mathcal{C})$ *the $(\infty, 1)$-category of locally constant factorization algebras on X with values in $(\mathcal{C}, \otimes)$* and $\mathrm{Fac}_X^{lc}(\mathcal{C})$ *its $(\infty, 1)$-subcategory of locally constant factorization algebras. We also denote* $\mathrm{coFac}_X(\mathcal{C})$ *(resp.* $\mathrm{coFac}_X^{lc}(\mathcal{C})$*) the $(\infty, 1)$-categories of (resp. locally constant) factorization coalgebras.*

In [GTZ2], we proved

Theorem 2.20 ([GTZ2, Theorem 6]). *The functor $(U, \mathcal{A}) \mapsto \int_U \mathcal{A}$ induces an equivalence of $(\infty, 1)$-categories between locally constant $N(\mathrm{Disk}(X))$-algebra and locally constant factorization algebras on the manifold X in the sense of* [CG]. *Further this functor is (equivalent to) the identity functor when restricted to open disks.*

This justifies our terminology of locally constant factorization algebras and factorization homology; further, the extension on any open set U of a (locally constant) $N(\mathrm{Disk}(X))$-algebra $\mathcal{A}$ is precisely given by the factorization homology $\int_U \mathcal{A}$, see *loc. cit.*.

Example 2.21 (Trivial example: the unit factorization algebra k). The unit object of the symmetric monoidal category of factorization algebras over a CW-complex or

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[8]an open cover of $\mathcal{U}$ is factorizing if, for all finite collections $x_1, \ldots, x_n$ of distinct points in U, there are pairwise disjoint open subsets $U_1, \ldots, U_k$ in $\mathcal{U}$ such that $\{x_1, \ldots, x_n\} \subset \bigcup_{i=1}^{k} U_i$

topological manifold with corners X is the trivial factorization algebra associated to the ground ring k. We review it here for latter use. We will simply denote it by k.

Its prefactorization algebra structure is given by $k(U) := k$ for any open set $U \subset X$ and the structure maps are given, for any pairwise disjoint open subsets $U_1, \ldots, U_r$ of an open $V \subset X$, by

$$k(U_1) \otimes \cdots \otimes k(U_r) = \bigotimes_{i=1}^{r} k \xrightarrow[\simeq]{\text{multiply}} k = k(V)$$

where the last map is the multiplication in the ring k.

Lemma 2.22. *The prefactorization algebra $U \mapsto k(U) = k$ is a factorization algebra. It is further naturally equivalent to the Hochschild chains (Proposition 3.9: $CH_U(k) \xrightarrow{\simeq} k(U) = k$*

k is by definition locally constant on any manifold and actually a commutative constant factorization algebra in the terminology of [GTZ2, § 4.2].

Proof of Lemma 2.22. The symmetry and associativity axioms of a prefactorization algebra follows respectively from the commutativity and associativity of the ring structure of k. Note that if X is a manifold, k is locally constant by definition and thus a factorization algebra. Note also that the Hochschild chain functor (as described in Section 3) induces a factorization algebra $U \mapsto CH_U(A)$ for any commutative algebra A by [GTZ2, Theorem 4]. For $A = k$, and any simplicial model $X_\bullet$ of a space X, one has that $CH_{X_\bullet}(k)$ is the differential graded commutative algebra associated to the constant simplicial k-module $n \mapsto CH_{X_n}^{simp}(k) \cong k$ see Definition 3.2. Hence the projection $CH_{X_\bullet} \twoheadrightarrow CH_{X_0}(k) = k$ is a natural (in $X_\bullet$) quasi-isomorphism of CDGA's and we obtain this way a natural (in X) equivalence $CH_X(k) \xrightarrow{\simeq} k$. The commutative diagram (induced by [GTZ2, Lemma 2])

$$
\begin{array}{ccc}
CH_V(k) & \xrightarrow{\simeq} & k \\
\mu_{U_1,\ldots,U_r,V} \Big\uparrow & & \Big\uparrow \text{multiply} \\
\bigotimes_{i=1}^{r} CH_{U_i}(k) & \xrightarrow{\simeq} & \bigotimes_{i=1}^{r} k
\end{array}
$$

proves that $U \mapsto k(U)$ is equivalent to $U \mapsto CH_U(k)$ and thus is a factorization algebra since the latter is (of course, the latter can also be checked by direct inspection rather easily). □

Remark 2.23 (Alternative definition: parametrized prefactorization algebras). There is a slight variation of the notion of locally constant (pre)factorization algebras, *i.e.*, locally constant $N(\text{Disk}(M))$-algebras. Following Lurie [L-HA, Remark 5.2.4.8], we let

Definition 2.24. $N(\text{Disk}(M)')$ be the ∞-operad associated to the colored operad $\text{Disk}(M)'$ whose objects are open embeddings $\mathbb{R}^n \hookrightarrow M$ and whose morphisms $\text{Disk}(M)'(\phi, \psi)$ are commutative diagrams

$$
\begin{array}{ccc}
\mathbb{R}^n & \xrightarrow{h} & \mathbb{R}^n \\
& \phi \searrow \quad \swarrow \psi & \\
& M &
\end{array}
$$

where f is an *open embedding*.

An $N(\mathrm{Disk}(M)')$-algebra $\mathcal{A}$ is said to be *locally constant* if the structure map $\mathcal{A}(h) : \mathcal{A}(\phi) \to \mathcal{A}(\psi)$ is a quasi-isomorphism for any open embedding h such that $\psi \circ h = \phi$ as in the above diagram.

Note that the (forgetful) functor $\iota : \phi \mapsto \phi(\mathbb{R}^n)$ is an equivalence of categories. Hence,

Proposition 2.25. *The functors $\iota^* : Disk(M)^{lc}\text{-}Alg \to Disk(M)'\text{-}Alg$ and $\iota^* : Fac_M^{lc} \to Disk(M)'^{lc}\text{-}Alg$ are equivalences of $(\infty, 1)$-categories.*

Similarly, the functors $\iota^* : Disk(M)^{lc}\text{-}coAlg \to Disk(M)'\text{-}Alg$ and $\iota^* : Fac_M^{lc} \to Disk(M)'^{lc}\text{-}coAlg$ are equivalences of $(\infty, 1)$-categories.

We will refer to locally constant $\mathrm{Disk}(M)'$-algebras as *locally constant parametrized factorization algebras.* By the above proposition 2.25, the two notions of factorizations algebras are essentially the same.

Unfolding Definition 2.24, we see that a locally constant parametrized factorization algebra $\mathcal{F}$ is thus a rule which associates to each embedding $\phi : \mathbb{R}^n \to M$ a chain complex $\mathcal{F}(\phi)$ with natural maps $\mathcal{F}(\phi_1) \otimes \cdots \otimes \mathcal{F}(\phi_r) \to \mathcal{F}(\psi)$ associated to any open embedding $h : \coprod_{i=1}^r \mathbb{R}^n \to \mathbb{R}^n$ such that $\psi \circ h = \coprod_{i=1}^r \phi_i : \coprod_{i=1}^r \mathbb{R}^n \to M$ (satisfying the obvious associativity and symmetry conditions). Further, for any $h : \phi \mapsto \psi$ (*i.e.* $\psi \circ h = \phi$), the structure map $\mathcal{F}(\phi) \to \mathcal{F}(\psi)$ is required to be a quasi-isomorphism.

Definition 2.26. Let U be an open subset of X. By restricting to open subsets of U, a (locally constant) factorization algebra $\mathcal{A}$ on X has a canonical restriction $\mathcal{A}_{|U} : Op(U) \ni V \mapsto \mathcal{A}(V)$ to a (locally constant) factorization algebra on U.

Factorization algebras satisfy a local-to-global property and can thus be defined out of their restriction to a basis or descent/gluing data. Indeed, let $\mathcal{U}$ be a *basis* for the topology of a manifold M which is stable by finite intersections and is also a factorizing cover (as in Remark 2.17).

Definition 2.27. A $\mathcal{U}$-*prefactorization algebra* is defined similarly to a prefactorization algebra on M, except that we are only considering opens that belongs to $\mathcal{U}$ in the definition. In other words, it is an algebra over $N\big(\mathrm{Disk}_{\mathcal{U}}(M)\big)$ where $\mathrm{Disk}_{\mathcal{U}}(M)$ is the colored sub-operad of $\mathrm{Disk}(M)$ whose only colors are those consisting of opens in $\mathcal{U}$.

Similarly, a $\mathcal{U}$-*factorization algebra* is defined similarly to a factorization algebra on M, except that we are only considering opens that belongs to $\mathcal{U}$ in the definition (in other words, it is a $\mathcal{U}$-prefactorization algebra satisfying the descent condition of Remark 2.17).

We refer to [CG, G2] for more details.

Proposition 2.28 (Costello-Gwilliam [CG])**.** *Let $\mathcal{F}$ be a $\mathcal{U}$-factorization algebra. There is an unique[9] factorization algebra $i_*^{\mathcal{U}}(\mathcal{F})$ on M extending $\mathcal{F}$[10] (that is equipped with a quasi-isomorphism of $\mathcal{U}$-factorization algebras $i_*^{\mathcal{U}}(\mathcal{F}) \to \mathcal{F}$).*

---

[9] up to natural equivalence

[10] More precisely, for an open set $V \subset X$, one has $i_*^{\mathcal{U}}(\mathcal{F})(V) := \check{C}(\mathcal{U}_V, \mathcal{F})$ where $\mathcal{U}_V$ is the open cover of V consisting of all open subsets of V which are in $\mathcal{U}$

In fact the canonical functor $Fac_{\mathcal{U}}(\mathcal{C}) \rightarrow Fac_M(\mathcal{C})$ is an equivalence of ∞-categories.

More generally, if $\mathcal{A}$ is a factorization algebra on X and $\mathcal{U} = (U_i)_{i \in I}$ is a cover of X, then $\mathcal{A}$ can be uniquely recovered by the data of the factorization algebras $\mathcal{A}_{|U_i}$ restricted to the U_i's (thanks to the Čech condition applied to suitable covers). In fact, any family of factorization algebras $\mathcal{F}_i$ on U_i, satisfying natural compatibility conditions on the intersections of the U_i's, extends uniquely into a factorization algebra on X; we refer to Costello-Gwilliam [CG, Section 4] (and [G2, § 5]) for details on this descent property of factorization algebras.

2.4. E_n-**algebras as factorization algebras.** Theorem 5.2.4.9 of [L-HA] (also see [Lu3, GTZ2]) provides an equivalence between E_n-algebras and *locally constant factorization algebra* on the open disk D^n:

Theorem 2.29. *Let $\mathcal{C}$ be a symmetric monoidal $(\infty, 1)$-category. There is a natural equivalence of $(\infty, 1)$-categories*

$$E_n\text{-}Alg(\mathcal{C}) \;\cong\; Fac_{\mathbb{R}^n}^{lc}(\mathcal{C}).$$

Similarly there is an equivalence between the $(\infty, 1)$-categories of locally constant factorization coalgebras on $\mathbb{R}^n$ and the one of E_n-coalgebras.

In particular, an E_n-algebra can be seen as an n-dimensional (topological) field theory (over the space-time manifold $\mathbb{R}^n$), providing an invariant for framed n-manifolds which is precisely computed by factorization homology.

Let X, Y be topological spaces and $f : X \rightarrow Y$ be continuous. There is a *pushforward functor*

(5) $$f_* : \mathrm{Fac}_X(\mathcal{C}) \rightarrow \mathrm{Fac}_Y(\mathcal{C})$$

that was constructed in [CG]. It is given, for $\mathcal{F} \in \mathrm{Fac}_X(\mathcal{C})$ and $V \in Op(Y)$, by $f_*(\mathcal{F})(V) = \mathcal{F}(f^{-1}(V))$. In particular, let $p : M \rightarrow pt$ be the canonical map from M to a point and $\mathcal{F}$ be a factorization algebra on M.

Notation: we also denote $p_*(\mathcal{F})$ the object $p_*(\mathcal{F})(pt) \cong \mathcal{F}(M)$. If $\mathcal{F}$ is locally constant, we have natural equivalences

(6) $$p_*(\mathcal{F}) \cong \mathcal{F}(M) \cong \int_M \mathcal{F}.$$

Assume X, Y are manifolds and let $\pi : X \times Y \rightarrow X$ be the canonical projection. Then, there is a factorization of the pushforward functor:

$$
\begin{array}{ccccc}
\mathrm{Fac}_{X \times Y}^{lc}(\mathcal{C}) & \xrightarrow{\;\pi_*\;} & \mathrm{Fac}_X^{lc}\big(\mathrm{Fac}_Y^{lc}(\mathcal{C})\big) & \xrightarrow{(Y \to pt)_*} & \mathrm{Fac}_X^{lc}(\mathcal{C}) \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{Fac}_{X \times Y}(\mathcal{C}) & \xrightarrow{\;\pi_*\;} & \mathrm{Fac}_X\big(\mathrm{Fac}_Y(\mathcal{C})\big) & \xrightarrow{(Y \to pt)_*} & \mathrm{Fac}_X(\mathcal{C})
\end{array}
$$

where the vertical arrows are induced by the fully faithfull inclusion of the locally constant factorization algebras inside all factorization algebras. The right horizontal arrows are induced by the pushforward along the canonical map $Y \rightarrow pt$. Here the fact that the pushforward of a locally constant factorization algebra is locally constant follows from the fact that the fibers are all naturally identified with the same manifold Y. We refer to [G2] for more details.

The fact that locally constant factorization algebras on $\mathbb{R}^n$ are E_n-algebras implies that, when $Y = \mathbb{R}^n$, the pushforward factors through a functor $\pi_* : Fac_{X \times \mathbb{R}^n}^{lc} \to Fac_X^{lc}(E_n\text{-Alg})$ see [GTZ2]. In particular, we can take $X = \mathbb{R}^m$. The following ∞-category version of Dunn's Theorem was proved by Lurie [L-HA] (and [GTZ2] for the pushforward interpretation):

Theorem 2.30 (Dunn's Theorem)**.** *There is an equivalence of $(\infty, 1)$-categories $E_{m+n}Alg \cong E_n\text{-}Alg(E_m\text{-}Alg)$.*

Under the equivalence $E_n\text{-}Alg \cong Fac_{\mathbb{R}^n}^{lc}$ (Theorem 2.29), the above equivalence is realized by the pushforward $\pi_* : Fac_{\mathbb{R}^m \times \mathbb{R}^n}^{lc} \to Fac_{\mathbb{R}^m}^{lc}(E_n\text{-}Alg)$ associated to the canonical projection $\pi : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^m$.

3. HIGHER HOCHSCHILD (CO)CHAINS FOR E_∞-ALGEBRAS

In this section we define and study higher Hochschild (co)chains modeled over spaces for E_∞-algebras with values in E_∞-modules.

3.1. Factorization homology of E_∞-algebras and higher Hochschild chains.

Factorization homology with values in E_∞-algebras has special properties. It becomes an homotopy invariant and can be defined over any space, providing an homology theory for spaces. Indeed, it identifies with the Hochschild chains modeled on a space and with values in an E_∞-algebra, see Theorem 3.13 below. We will denote $(X, A) \mapsto CH_X(A)$ the latter construction which we explain further in this section. The reader familiar with factorization homology for commutative algebras can skip it and keep in mind that $CH_X(A)$ means factorization homology extended to spaces.

The Hochschild chains $CH_{X_\bullet, \bullet}(A, A)$ over a simplicial set $X_\bullet$ with value in a CDGA A is defined in [P]. As explained in [GTZ2], it can be defined using the PROPic definition of commutative (differential graded) k-algebras as follows. A CDGA over k is a strict symmetric monoidal functor $A : Fin \to k\text{-Mod}$ from the category of finite sets (with disjoint union for the monoidal structure) to the category of chain complexes (with tensor product as the monoidal structure). Given a finite simplicial set $\Delta^{op} \to Fin$ we can compose these two functors to get a simplicial k-module. The geometric realization of this simplicial k-module is the Hochschild complex modeled on X. In fact one can do better. A strictly symmetric monoidal functor $A : Fin \to k\text{-Mod}$ has a canonical lift to $A : Fin \to k\text{-Alg}$, along the forgetful functor $k\text{-Alg} \to k\text{-Mod}$. This is due the simple fact that for a commutative algebra A, the multiplication $m : A \otimes A \to A$ is a map of algebras or, said otherwise, the fact that the tensor product is a coproduct in CDGA. Thus, the above procedure gives rise to a simplicial CDGA whose geometric realization is the Hochschild complex modeled on $X_\bullet$, with a canonical CDGA structure. In the classical example $X = S^1$, whereby we get the classical Hochschild complex, this CDGA structure is given by the shuffle product on $CH_\bullet(A, A)$. Another way of saying this is to say that that $CDGA$ is tensored over simplicial sets and that Hochschild chains over $X_\bullet$ with values in A is simply the tensor $A \boxtimes X_\bullet$ (see [L-HA, L-HTT, EKMM, Ke, MCSV] for details on tensored categories over spaces and Remark 2.7). We now discuss how all of the above generalizes to the case of E_∞-algebras.

Recall from Section 2 that the $(\infty, 1)$-category of E_∞-algebras (with value in chain complexes) is equivalent to the $(\infty, 1)$-category $Fun^\otimes(N(Fin), k\text{-Mod}_\infty)$ of

(lax) monoidal functors from (the ∞-category associated to) Fin to the $(\infty, 1)$-category of chain complexes (see Lurie [Lu2, L-HA, KM]). We denote

$$\left(X \mapsto CH_X^{simp}(A)\right) \in Fun^{\otimes}(N(Fin), k\text{-Mod}_{\infty})$$

the[11] monoidal functor associated to an E_{∞}-algebra A. This functor extends naturally to a functor $Fun^{\otimes}(N(Set), k\text{-Mod}_{\infty})$ (where Set is the category of sets) by taking colimits; that is to say, $X \mapsto \varinjlim_{Fin \ni K \to X} CH_K^{simp}(A)$.

By Proposition 2.9, $CH_K^{simp}(A)$ has a natural structure of E_{∞}-algebras; more precisely, the functor $\left(X \mapsto CH_X^{simp}(A)\right)$ factors as

$$N(Fin) \longrightarrow E_{\infty}\text{-Alg} \xrightarrow{forget} k\text{-Mod}_{\infty}.$$

We simply denote $\left(X \mapsto CH_X^{simp}(A)\right) \in Fun^{\otimes}(N(Set), E_{\infty}\text{-Alg})$ the induced lift.

Remark 3.1. Fixing a set X, $CH_X(A)$ is (functorially) quasi-isomorphic to the tensor product $A^{\otimes X}$ (where A is viewed as a chain complex). Note that this construction (of the underlying chain complex structure) is the same as the one in[GTZ, Section 2.1] in the case of CDGAs. However the functorial structure involves higher homotopies and not only the multiplication and seems difficult to write explicitly on this particular choice of cochain complex.

Let $DK : sk\text{-Mod}_{\infty} \to k\text{-Mod}_{\infty}$ be the Dold-Kan functor from the $(\infty, 1)$-category of simplicial k-modules to the chain complexes. The Dold-Kan functor refines to a functor $sE_{\infty}\text{-Alg} \to E_{\infty}\text{-Alg}$ from simplicial E_{∞}-algebras to differential graded E_{∞}-algebras which preserves weak-equivalences (see [M1, Section 3]).

Definition 3.2. The derived Hochschild chains of an E_{∞}-algebra A and a simplicial set $X_{\bullet}$ is

$$CH_{X_{\bullet}}(A) := DK\Big(\varinjlim_{Fin \ni K \to X_{\bullet}} CH_K^{simp}(A)\Big).$$

Remark 3.3. In the case where the E_{∞}-algebra A is strict, *i.e.* a CDGA, it follows from Corollary 3.7 below that $CH_{X_{\bullet}}(A)$ is quasi-isomorphic to the Hochschild chain complex over $X_{\bullet}$ described in details in[GTZ, Section 2.1] (also see[P, G1, GTZ2]).

Proposition 3.4. *The derived Hochschild chain* $(X_{\bullet}, A) \mapsto CH_{X_{\bullet}}(A)$ *lifts as a functor of* $(\infty, 1)$-*categories*

$$CH : sSet_{\infty} \times E_{\infty}\text{-Alg} \longrightarrow E_{\infty}\text{-Alg}.$$

Further, it is the tensor of A and $X_{\bullet}$ in E_{∞}-Alg, i.e., *there is a natural equivalence* $CH_{X_{\bullet}}(A) \cong A \boxtimes X_{\bullet}$. *In particular,*

$$(7) \qquad Map_{sSet_{\infty}}\big(X_{\bullet}, Map_{E_{\infty}\text{-Alg}}(A, B)\big) \cong Map_{E_{\infty}\text{-Alg}}(CH_{X_{\bullet}}(A), B).$$

Note that we could also just have used the tensor definition $A \boxtimes X_{\bullet}$ to define higher Hochschild chains.

Proof of Proposition 3.4. Proposition 3.6 below implies that the derived Hochschild chain functor is invariant under (weak) equivalences of E_{∞}-algebras and simplicial sets and thus lifts as an $((\infty, 1)\text{-})$functor $sSet_{\infty} \times E_{\infty}\text{-Alg} \to k\text{-Mod}_{\infty}$, $(X_{\bullet}, A) \mapsto CH_{X_{\bullet}}(A)$. Since the tensor products of E_{∞}-algebras is an E_{∞}-algebra, $CH_{K_{\bullet}}^{simp}(A)$

---

[11]it is unique up to contractible choices

is a simplicial E_∞-algebra for any simplicial set $K_\bullet$. Since the (refined) Dold-Kan functor $sE_\infty\text{-Alg} \to E_\infty\text{-Alg}$ preserves weak-equivalences [M1], the derived Hochschild chain functor lifts as a functor of $(\infty, 1)$-categories $CH : sSet_\infty \times E_\infty\text{-Alg} \to E_\infty\text{-Alg}$. By Proposition 2.8, $CH_{X_\bullet}(A) \cong A \boxtimes X_\bullet$ in $sE_\infty\text{-Alg}$ from which the second assertion of the Proposition follows after passing to geometric realization. $\square$

Remark 3.5. There is a derived functor interpretation of the above Definition 3.2. Recall that to any simplicial set $X_\bullet$, one can associate a canonical E_∞-coalgebra structure on its chains [Ma, BF], denoted $C_*(X_\bullet)$ (Example 2.6). Dually to the case of algebras, an E_∞-coalgebra C defines a *contravariant* monoidal functor $X \mapsto CH^{simp\ co}_X(C)$, *i.e.*, an object of $Fun^\otimes(N(Fin)^{op}, k\text{-Mod}_\infty)$.

In particular, an E_∞-coalgebra C defines a *right* module over the ∞-operad $\mathbb{E}_\infty^\otimes$ and an E_∞-algebra a *left* module over the ∞-operad $\mathbb{E}_\infty^\otimes$. We can thus form their (derived) tensor products $C \overset{\mathbb{L}}{\underset{\mathbb{E}_\infty^\otimes}{\otimes}} A$ which is computed as a (homotopy) coequalizer:

$$C \overset{\mathbb{L}}{\underset{\mathbb{E}_\infty^\otimes}{\otimes}} A \cong hocoeq \Big(\coprod_{f:\{1,\dots,q\}\to\{1,\dots,p\}} C^{\otimes p} \otimes \mathbb{E}_\infty^\otimes(q,p) \otimes A^{\otimes q} \rightrightarrows \coprod_n C^{\otimes n} \otimes A^{\otimes n} \Big)$$

where the maps $f : \{1, \dots, q\} \to \{1, \dots, p\}$ are maps of sets. The upper map in the coequalizer is induced by the maps $f^* : C^{\otimes p} \otimes \mathbb{E}_\infty^\otimes(q,p) \otimes A^{\otimes q} \to C^{\otimes q} \otimes A^{\otimes q}$ obtained from the coalgebra structure of C and the lower map is induced by the maps $f_* : C^{\otimes p} \otimes \mathbb{E}_\infty^\otimes(q,p) \otimes A^{\otimes q} \to C^{\otimes p} \otimes A^{\otimes p}$ induced by the algebra structure.

Proposition 3.6. *Let $X_\bullet$ be a simplicial set and A be an E_∞-algebra. There is a natural equivalence*

$$CH_{X_\bullet}(A) \cong C_*(X_\bullet) \overset{\mathbb{L}}{\underset{\mathbb{E}_\infty^\otimes}{\otimes}} A$$

$$\cong hocoeq \Big(\coprod_{f:\{1,\dots,q\}\to\{1,\dots,p\}} C_*(X_\bullet)^{\otimes p} \otimes \mathbb{E}_\infty^\otimes(q,p) \otimes A^{\otimes q} \rightrightarrows \coprod_n C_*(X_\bullet)^{\otimes n} \otimes A^{\otimes n} \Big)$$

Proof. Note that the E_∞-coalgebra structure on $C_*(X_\bullet)$ is given by the functor $N(Fin_*)^{op} \to k\text{-Mod}_\infty$ defined by $I \mapsto k[Hom_{Fin}(I, X_\bullet)]$. The rest of the proof now is the same as in [GTZ2, Proposition 4]. $\square$

In [GTZ2], a functor $CH^{cdga} : sSet_\infty \times CDGA_\infty \to CDGA_\infty$ was defined[12]. There is a forgetful functor $CDGA_\infty \to E_\infty\text{-Alg}$. Proposition 3.6, Proposition 4 in [GTZ2] and the equivalence $\mathbb{E}_\infty^\otimes \overset{\cong}{\to} Comm^\otimes$ yield

Corollary 3.7. *If A is a commutative differential graded algebra, the following diagram is commutative in the $(\infty, 1)$-category $Fun(sSet_\infty \times CDGA_\infty, E_\infty\text{-Alg})$:*

$$\begin{array}{ccc} sSet_\infty \times CDGA_\infty & \xrightarrow{CH^{cdga}} & CDGA_\infty \\ \downarrow & & \downarrow \\ sSet_\infty \times E_\infty\text{-}Alg & \xrightarrow{CH} & E_\infty\text{-}Alg \end{array}$$

In particular, $CH^{cdga}_{X_\bullet}(A)$ is naturally equivalent to $CH_{X_\bullet}(A)$.

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[12]it was simply denoted CH in loc. cit.

In other words, the corollary means that the functors CH^{cdga} and CH are equivalent (for a CDGA).

Remark 3.8. In the sequel we will use the equivalence given by corollary 3.7 to identify the functors CH and CH^{cdga} without further notice.

There is an equivalence of $(\infty, 1)$-categories $sSet_\infty \overset{\sim}{\rightleftarrows} Top_\infty$ induced by the underlying Quillen equivalence between $sSet$ and Top $\overset{\sim}{[GJ, Ho]}$. The left and right equivalences are respectively induced by the standard singular set functor $X \mapsto S_\bullet(X) := Map(\Delta^\bullet, X)$ and geometric realization $X_\bullet \mapsto |X_\bullet|$ functors. In particular, we can replace simplicial sets by *topological spaces* in Definition 3.2 and Proposition 3.6 to get the following analogue of Proposition 3.4. Letting $C_*(X)$ be the natural E_∞-coalgebra structure on the singular chains of X, we deduce from Proposition 3.4 and Proposition 3.6:

Proposition 3.9. *The derived* Hochschild chain with value in an E_∞-algebra A modeled on a space X *given by*

$$CH_X(A) \quad := \quad DK\Big(\varinjlim_{Fin \ni K \to S_\bullet(X)} CH_K^{simp}(A) \Big)$$

$$\cong \quad C_*(X) \overset{\mathbb{L}}{\underset{\mathbb{E}_\infty^\otimes}{\otimes}} A$$

induces a $(\infty, 1)$-*functor* $CH : (X, A) \mapsto CH_X(A)$ *from* $Top_\infty \times E_\infty$-*Alg to* E_∞-*Alg. Further, one has a natural equivalence* $A \boxtimes X \cong CH_X(A)$; *in particular*

$$(8) \qquad Map_{Top_\infty}\big(X, Map_{E_\infty\text{-}Alg}(A, B)\big) \cong Map_{E_\infty\text{-}Alg}(CH_X(A), B).$$

Remark 3.10. Since $(X, A) \mapsto CH_X(A) \cong A \boxtimes X$ is a functor of both variables, $CH_X(A)$ has a natural action of the topological monoid $Map_{Top_\infty}(X, X)$ (and in particular of the group $Homeo(X)$). This means that there is a monoid map $Map_{Top_\infty}(X, X) \to Map_{E_\infty\text{-}Alg}(CH_X(A), CH_X(A))$; in other words a chain map $C_*\big(Map_{Top_\infty}(X, X)\big) \otimes CH_X(A) \to CH_X(A)$ which makes $CH_X(A)$ a $Map_{Top_\infty}(X, X)$-algebra in E_∞-Alg (for the monad associated to the monoid $Map_{Top_\infty}(X, X)$).

Similarly, given any map $f : X \times K \to Y$ of topological spaces, we get a canonical map $\tilde{f} : K \to Map_{E_\infty\text{-}Alg}(CH_X(A), CH_Y(A))$ in Top_∞ as follows. By Proposition 3.9, the map $f_* : CH_{X \times K}(A) \to CH_Y(A)$ yields a natural map of mapping spaces:

$$(9) \quad \tau_f : Map_{E_\infty\text{-}Alg}\big(CH_Y(A), CH_Y(A)\big)$$
$$\xrightarrow{(f_*)^*} Map_{E_\infty\text{-}Alg}\big(CH_{X \times K}(A), CH_Y(A)\big)$$
$$\cong Map_{Top_\infty}\big(K, Map_{E_\infty\text{-}Alg}(CH_X(A), CH_Y(A))\big)$$

where the last equivalence follows from Proposition 3.9 (8) and Corollary 3.29.(4) below. Applying the map (9) to the identity $id_{CH_Y(A)}$ we get the map

$$(10) \qquad \tilde{f} := \tau_f(id_{CH_Y(A)}) \in Map_{Top_\infty}\big(K, Map_{E_\infty\text{-}Alg}(CH_X(A), CH_Y(A))\big).$$

In particular, the map $\tilde{f}$ yields a map

$$(11) \qquad\qquad\qquad \tilde{f}_* : C_*(K) \otimes CH_X(A) \to CH_Y(A)$$

in $k\text{-Mod}_\infty$.

The map $\tilde{f}$ (respectively $\tilde{f}_*$) are functorial in the obvious sense. Indeed, let $f : K \times X \to Y$, $g : K \times Y \to Z$, $j : L \to K$ be continuous maps. Out of f and g we can form the map

$$K \times X \xrightarrow{(p_K, f)} K \times Y \xrightarrow{g} Z$$

where $p_K : K \times X \to K$ is the canonical projection while out of j and f, we can form the composition

$$L \times X \xrightarrow{j \times id_X} K \times X \xrightarrow{f} Y.$$

We thus get the maps $\tilde{f}$, $\tilde{g}$ as well as

$$(f \circ \widetilde{(j \times id_X)}) : L \to Map_{E_\infty\text{-Alg}}(CH_X(A), CH_Y(A)))$$

and $(g \circ \widetilde{(p_K, f)})) : K \to Map_{E_\infty\text{-Alg}}(CH_X(A), CH_Z(A)))$. The functoriality relation are given by:

Proposition 3.11. *The following two diagrams*

$$K \xrightarrow{(\tilde{f}, \tilde{g})} Map_{E_\infty\text{-}Alg}(CH_X(A), CH_Y(A)) \times Map_{E_\infty\text{-}Alg}(CH_Y(A), CH_Z(A)) ,$$

with vertical arrow $-\circ-$ to $Map_{E_\infty\text{-}Alg}(CH_X(A), CH_Z(A))$ and diagonal $(g \circ \widetilde{(p_K, f)}))$

where the vertical arrow is the composition of morphisms, and

$$L \xrightarrow{(f \circ \widetilde{(j \times id_X)}))} Map_{E_\infty\text{-}Alg}(CH_X(A), CH_Y(A))$$

with $j : L \to K$ and diagonal $\tilde{f}$

are commutative.

Proof. The result follows from the following two factorizations

$$(f \circ (j \times id_X))_* : CH_{L \times X}(A) \xrightarrow{(j \times id_X)_*} CH_{K \times X}(A) \xrightarrow{f_*} CH_Y(A),$$

$$(g \circ (p_K, f))_* : CH_{K \times X}(A) \xrightarrow{(p_K \times f)_*} CH_{K \times Y}(A) \xrightarrow{g_*} CH_Z(A)$$

which in turn follow from Proposition 3.9. $\qquad\square$

Example 3.12. Consider a commutative diagram of spaces

$$\begin{array}{ccc} L \times K \times X & \xrightarrow{p_L \times f} & L \times Y \\ \downarrow{\scriptstyle (q \times id_X)} & & \downarrow{\scriptstyle g} \\ R \times X & \xrightarrow{h} & Z \end{array}$$

where $p_L : L \times K \times X \to L$ is the canonical projection, $f : K \times X \to Y$ and $q : L \times K \to R$ are continuous maps. Then by Proposition 3.11, we get that the

following diagram is commutative

$$\begin{array}{ccc} C_*(L \times K) \otimes CH_X(A) & \xrightarrow{\tilde{f}_*} & C_*(L) \otimes CH_Y(A) \\ {\scriptstyle q_* \otimes id} \downarrow & & \downarrow {\scriptstyle \tilde{g}_*} \\ C_*(R) \otimes CH_X(A) & \xrightarrow{\tilde{h}_*} & CH_Z(A). \end{array}$$

As we previously mentioned, the higher Hochschild functor (modeled on spaces) agrees with factorization homology (see [L-HA, F1] and Definition 2.13) for E_∞-algebras. Indeed the following result (whose CDGA version was proved in [GTZ2]) was proved by Francis [F1].

Theorem 3.13. *Let M be a manifold of dimension m and A be an E_∞-algebra viewed as an $N(Disk(M))$-algebra (by restriction of structure, Example 2.16). Then, the factorization homology $\int_M A$ of M with coefficients in A is naturally equivalent to $CH_M(A)$.*

Proof. The proof is the same as the ones for CDGA's in [GTZ2] (see Theorem 6 and Corollary 9 in *loc. cit.*) using the axioms of Theorem 3.28. Further, as pointed out by John Francis [F1], the proof also applies to topological manifolds. □

In particular, it follows that the factorization homology of an E_∞-algebra and framed manifold M is canonically an E_∞-algebra which is independent of the choices of framing, and further, can be extended functorially with respect to all continuous maps $h : N \to M$.

Remark 3.14. There is also a nice interpretation of Hochschild chains over spaces in terms of derived algebraic geometry. Let $\mathbf{dSt}_k$ be the (model) category of derived stacks over the ground ring k described in details in [TV, Section 2.2]. This category, which admits internal Hom's denoted by $\mathbb{R} Map(F, G)$ following [TV, TV2], is an enrichment of the homotopy category of spaces. Indeed, any simplicial set $X_\bullet$ yields a constant simplicial presheaf E_∞-Alg $\to sSet$ defined by $R \mapsto X_\bullet$ which, in turn, can be stackified. We denote $\mathfrak{X}$ the associated stack, *i.e.* the stackification of $R \mapsto X_\bullet$, which depends only on the (weak) homotopy type of $X_\bullet$. It is sometimes called the *Betti stack* of $X_\bullet$.

For a (derived) stack $\mathfrak{Y} \in \mathbf{dSt}_k$, we denote $\mathcal{O}_\mathfrak{Y}$ its functions, *i.e.*, $\mathcal{O}_\mathfrak{Y} := \mathbb{R}\underline{Hom}(\mathfrak{Y}, \mathbb{A}^1)$, (see [TV]).

Corollary 3.15. *Let $\mathfrak{R} = \mathbb{R} Spec(R)$ be an affine derived stack (for instance an affine stack) [TV] and $\mathfrak{X}$ be the stack associated to a space X. Then the Hochschild chains over X with coefficients in R represent the mapping stack $\mathbb{R} Map(\mathfrak{X}, \mathfrak{R})$. That is, there are canonical equivalences*

$$\mathcal{O}_{\mathbb{R} Map(\mathfrak{X}, \mathfrak{R})} \cong CH_X(R), \qquad \mathbb{R} Map(\mathfrak{X}, \mathfrak{R}) \cong \mathbb{R} Spec\big(CH_X(R)\big)$$

Proof. The proof is analogous to the one of [GTZ2, Corollary 6.4.4]. □

Note that if a group G acts on X, the natural action of G on $CH_X(A)$ (see Remark 3.10) identifies with the natural one of G on $\mathbb{R} Map(\mathfrak{X}, \mathfrak{R})$ under the equivalence given by Corollary 3.15.

3.2. **Higher Hochschild (co)chains with values in E_∞-modules.** We now consider a dual notion of the Hochschild chain functor, which is well defined in the E_∞-case.

Let $\epsilon : pt \to X_\bullet$ be a base point of $X_\bullet$. The map ϵ yields a map of E_∞-algebras $A \cong CH_{pt}(A) \xrightarrow{\epsilon_*} CH_{X_\bullet}(A)$ and thus makes $CH_{X_\bullet}(A)$ an A-module. Let M be another E_∞-A-module.

Definition 3.16. The (derived) *Hochschild cochains* of an E_∞-algebra A with value in M over (the pointed simplicial set) $X_\bullet$ is given by

$$CH^{X_\bullet}(A, M) = Hom_A(CH_{X_\bullet}(A), M),$$

the (derived) chain complex of the underlying left E_1-A-module homomorphisms.

The definition above depends on the choice of the base point even though we do not write it explicitly in the definition. We define similarly $CH^X(A, M)$ for any *pointed topological space X*.

Remark 3.17. According to Theorem 5.13, one can also alternatively consider the chain complex of E_∞-A-modules in Definition 3.16.

Definition 3.18. The *Hochschild chains* of an E_∞-algebra A with values in M over (the pointed simplicial set) $X_\bullet$ is defined as

$$CH_{X_\bullet}(A, M) = M \overset{\mathbb{L}}{\underset{A}{\otimes}} CH_{X_\bullet}(A)$$

the relative tensor product of E_∞-A-modules (as defined, for instance, in [L-HA, Section 3.3.3] or [KM]).

Remark 3.19. Any E_∞-A-module has an underlying E_1-module structure given by the forgetful functor $A\text{-}Mod^{E_\infty} \to A\text{-}Mod^{E_1}$ hence both a left and right A-module structure. Thus, given two E_∞-A-modules M, N, one can form their relative tensor product $M \otimes_A^{\mathbb{L}} N$ where M is viewed as a right A-module, N as a left A-module and A as an E_1-algebra. According to Theorem 5.13 and [L-HA, Section 4.4.1] or [KM, Section 5], this tensor is equivalent (as an object of $k\text{-}Mod_\infty$) to the relative tensor product computed in E_∞-A-modules. Hence, the tensor product of Definition 3.18 can be computed using this alternative definition.

Since the based point map $\epsilon_* : A \to CH_{X_\bullet}(A)$ is a map of E_∞-algebras, the canonical module structure of $CH_{X_\bullet}(A)$ over itself induces a module structure on $CH_{X_\bullet}(A, M)$ over $CH_{X_\bullet}(A)$ after tensoring by A (also see [KM, Part V], [L-HA]):

Lemma 3.20. *Let M be in $A\text{-}Mod^{E_\infty}$, that is, M is an E_∞-A-module. Then $CH_{X_\bullet}(A, M)$ is canonically a $CH_{X_\bullet}(A)$-E_∞-module.*

Remark 3.21. By definition, if A is endowed with its canonical A-E_∞-module structure, the natural map $CH_{X_\bullet}(A, A) \cong A \otimes_A^{\mathbb{L}} CH_{X_\bullet}(A) \to CH_{X_\bullet}(A)$ is an equivalence of $CH_{X_\bullet}(A)$-modules. Hence, tensoring by $M \otimes_A -$, we get a canonical lift of the relative tensor products $M \otimes_A^{\mathbb{L}} CH_{X_\bullet}(A)$, computed as a relative tensor product of left and right modules over A seen as an E_1-algebra, to a $CH_{X_\bullet}(A)$-E_∞-module as well.

Proposition 3.22. *The derived Hochschild chain $CH_{X_\bullet}(A, M)$ with value in an E_∞-algebra A and an A-module M over a space $X_\bullet$ given by Definition 3.18 induces*

a functor of $(\infty, 1)$-categories $CH : (X_\bullet, M) \mapsto CH_{X_\bullet}(\iota(M), M)$ from $sSet_{*\infty} \times Mod^{E\infty}$ to $Mod^{E\infty}$.

The derived Hochschild cochains $CH^{X_\bullet}(A, M)$ with value in an E_∞-algebra A and an A-module M over a space $X_\bullet$ given by Definition 3.16 induces a functor of $(\infty, 1)$-categories $(X_\bullet, M) \mapsto CH^{X_\bullet}(A, M)$ from $sSet_{*\infty}^{op} \times A\text{-}Mod^{E\infty}$ to $A\text{-}Mod^{E\infty}$, which is further contravariant[13] with respect to A.

Proof. It follows from Lemma 3.20 and § 3.1. The fact that homomorphisms of A-E_∞-modules have a canonical structure of A-E_∞-modules follows from the same argument as for the tensor product above or from [KM, Theorem V.8.1]. □

Remark 3.23. As usual, one obtains a similar version of the above Definition 3.18 and Lemma 3.20 for pointed topological space X.

Remark 3.24. If A is a CDGA and M a left A-module, similarly to Corollary 3.7, there are natural equivalences

$$CH^{cdga}_{X_\bullet}(A, M) \cong CH_{X_\bullet}(A, M), \qquad CH^{X_\bullet}_{cdga}(A, M) \cong CH^{X_\bullet}(A, M)$$

where $CH^{cdga}_{X_\bullet}(A, M)$ and $CH^{X_\bullet}_{cdga}(A, M)$ are the usual higher Hochschild chain and cochain functors for CDGA's and their modules defined respectively in [P] and [G1].

3.3. **Axiomatic characterization.** The axiomatic approach to Hochschild functors over spaces for CDGA's studied in the authors' previous work [GTZ2] extends formally to E_∞-algebras as well. It is actually an immediate corollary of the fact that E_∞-Alg (as well as any presentable $(\infty, 1)$-category) is tensored over simplicial sets in a unique way (up to homotopy). We now recall quickly the axiomatic characterization (similar to the Eilenberg-Steenrod axioms) and some consequences for Hochschild theory over spaces with value in $Mod^{E\infty}$. A similar story for factorization homology of E_n-algebras has been developed recently by Francis [F2, AFT].

We first collect the axioms characterizing the (derived) Hochschild chain theory over spaces into the following definition. Let $Forget : Top_{*\infty} \to Top_\infty$ be the functor that forget the base point.

Definition 3.25. An E_∞-homology theory[14] is a pair of ∞-functors $\mathcal{CA} : Top_\infty \times E_\infty\text{-Alg} \to E_\infty\text{-Alg}$, denoted $(X, A) \mapsto \mathcal{CA}_X(A)$, and $\mathcal{CM} : Top_{*\infty} \times Mod^{E\infty} \to Mod^{E\infty}$, denoted $(X, M) \mapsto \mathcal{CM}_X(M)$, fitting in a commutative diagram

(12)
$$
\begin{array}{ccc}
Top_{*\infty} \times Mod^{E\infty} & \xrightarrow{\;\mathcal{CM}\;} & Mod^{E\infty} \\
{\scriptstyle Forget \times \iota}\big\downarrow & & \big\downarrow{\scriptstyle \iota} \\
Top_\infty \times E_\infty\text{-Alg} & \xrightarrow{\;\mathcal{CA}\;} & E_\infty\text{-Alg}
\end{array}
$$

satisfying the following axioms:

   i) **value on a point:** there is a natural equivalence $\mathcal{CM}_{pt}(M) \cong M$ in $Mod^{E\infty}$;
   ii) **monoidal:** the natural map

$$\mathcal{CM}_X(M) \otimes \mathcal{CA}_Y(\iota(M)) \xrightarrow{\;\simeq\;} \mathcal{CM}_{X \coprod Y}(M)$$

   (where $X \in Top_{*\infty}$ and $Y \in Top_\infty$) is an equivalence.

---

[13]using the canonical functor (similar to the one of Example 2.2) $f_* : B\text{-}Mod^{E\infty} \to A\text{-}Mod^{E\infty}$ associated to any E_∞-algebras map $f : A \to B$

[14]with values in the symmetric monoidal $(\infty, 1)$-category $(k\text{-Mod}_\infty, \otimes)$

iii) excision: $\mathcal{CM}$ commutes with homotopy pushout of spaces, *i.e.*, there is a canonical equivalence

$$\mathcal{CM}_{X \cup_Z^h Y}(M) \cong \mathcal{CM}_X(M) \overset{\mathbb{L}}{\underset{\mathcal{CA}_Z(\iota(M))}{\otimes}} \mathcal{CA}_Y(\iota(M))$$

where $X \in Top_{*\infty}$, $Y, Z \in Top_\infty$.

Remark 3.26. Since any E_∞-algebra is canonically a module over itself, there is also a canonical functor $\phi : E_\infty\text{-Alg} \to Mod^{E_\infty}$, hence a functor $\left(- \coprod\{*\}\right) \times \phi : Top_\infty \times E_\infty\text{-Alg} \to Top_{*\infty} \times Mod^{E_\infty}$ giving rise, by composition with $\iota \circ \mathcal{CM}$ to a functor $\psi : Top_\infty \times E_\infty\text{-Alg} \to E_\infty\text{-Alg}$. By axioms $i)$ and $ii)$ in Definition 3.25 and commutativity of the diagram (12), we get a natural equivalence

$$\psi_X(A) \; \cong \; \phi(A) \otimes \mathcal{CA}_X(A).$$

Hence, the functor $\mathcal{CA}$ is actually completely defined by the functor $\mathcal{CM}$.

Remark 3.27. We also define a generalized E_∞-homology theory to be a triple of functors $F : Mod^{E_\infty} \to Mod^{E_\infty}$, $\mathcal{CA} : Top_\infty \times E_\infty\text{-Alg} \to E_\infty\text{-Alg}$ and $\mathcal{CM} : Top_{*\infty} \times Mod^{E_\infty} \to Mod^{E_\infty}$ satisfying all properties as in Definition 3.25 except that the value on a point axiom is modified by requiring a natural equivalence $\mathcal{CM}_{pt}(M) \cong F(M)$ in Mod^{E_∞}.

The next theorem shows that higher Hochschild homology theory is the unique functor satisfying the assumptions of Definition 3.25.

Theorem 3.28. (1) *The derived Hochschild chains functors $CH : Top_\infty \times E_\infty\text{-Alg} \to E_\infty\text{-Alg}$ (see Proposition 3.9) and the derived Hochschild chains with value in a module $CH_X : Top_{*\infty} \times Mod^{E_\infty} \to Mod^{E_\infty}$ (see Proposition 3.22) form a E_∞-homology theory in the sense of Definition 3.25.*
(2) *Any E_∞-homology theory (in the sense of Definition 3.25) is naturally equivalent to derived Hochschild chains, i.e., there are natural equivalences $\mathcal{CA}_X(A) \cong CH_X(A)$ and $\mathcal{CM}_X(M) \cong CH_X(\iota(M), M)$.*

Proof. This is essentially implied by the fact that $CH_X(A) \cong A \boxtimes X$ is the tensor of A with the space X and that such a tensor is defined uniquely, see [L-HTT, Corollary 4.4.4.9]. Note that the first assertion follows from Proposition 3.22 and Proposition 3.9. The proof of the uniqueness follows from the proofs of Theorem 4.2.7 and Theorem 4.3.1 in [GTZ2]. The excision and the value on a point axioms applied to $X = Z = pt$ show that there is a natural equivalence

$$\mathcal{CM}_Y(M) \cong M \overset{\mathbb{L}}{\underset{\iota(M)}{\otimes}} \mathcal{CA}_Y(\iota(M))$$

which reduces to proving the assertion for $\mathcal{CA}$. Since $\iota : Mod^{E_\infty} \to E_\infty\text{-Alg}$ is monoidal, $\mathcal{CA}$ is monoidal. Similarly, the natural equivalence(4) implies that $\mathcal{CA}$ satisfies the excision axiom (in the category of E_∞-algebras). Now the proof of [GTZ2, Theorem 2] applies verbatim. The argument boils down to the fact that Top_∞ is generated by a point using coproducts and homotopy pushouts. $\qquad \square$

We now list a few easy properties derived from the above Theorem 3.28.

Corollary 3.29. (1) *The derived Hochschild chain functor is the unique functor $Top_\infty \times E_\infty\text{-Alg} \to E_\infty\text{-Alg}$ satisfying the following three axioms*

(a) **value on a point:** *There is a natural equivalence of E_∞-algebras* $CH_{pt}(A) \cong A$.

(b) **coproduct:** *There are canonical equivalences*

$$CH_{\coprod_I (X_i)}(A) \xleftarrow{\simeq} \varinjlim_{\substack{K \subset I \\ K \text{ finite}}} \bigotimes_{k \in K} CH_{X_k}(A)$$

(c) **homotopy glueing/pushout:** *there are natural equivalences*

$$CH_{X \cup^h_Z Y}(A) \xleftarrow{\simeq} CH_X(A) \otimes^{\mathbb{L}}_{CH_Z(A)} CH_Y(A).$$

(2) *(generalized uniqueness)* *Let* $F : Mod^{E_\infty} \to Mod^{E_\infty}$, $\mathcal{CA} : Top_\infty \times E_\infty\text{-}Alg \to E_\infty\text{-}Alg$ *and* $\mathcal{CM} : Top_{*\infty} \times Mod^{E_\infty} \to Mod^{E_\infty}$ *be a generalized* E_∞-*homology theory. Then there is a natural equivalence*

$$\mathcal{CM}_X(M) \cong CH_X\big(\iota(F(M)), F(M)\big).$$

(3) *(commutations with colimits)* *The derived Hochschild chains functors* $CH : Top_\infty \times E_\infty\text{-}Alg \to E_\infty\text{-}Alg$ *and* $CH : Top_{*\infty} \times Mod^{E_\infty} \to Mod^{E_\infty}$ *commutes with finite colimits in* Top_∞ *and all colimits in* Mod^{E_∞}, *that is there are natural equivalences*

$$CH_{\varinjlim_{\mathcal{F}} X_i}(\iota(M), M) \cong \varinjlim_{\mathcal{F}} CH_{X_i}(\iota(M), M) \quad \text{(for a finite category } \mathcal{F}\text{)},$$

$$CH_X(\varinjlim A_i) \cong \varinjlim CH_X(A_i).$$

(4) **(product)** *Let* X, Y *be pointed spaces,* $M \in Mod^{E_\infty}$ *and* $A = \iota(M) \in E_\infty\text{-}Alg$. *There is a natural equivalence*

$$CH_{X \times Y}(A, M) \xrightarrow{\sim} CH_X\left(CH_Y(A), CH_Y(A, M)\right)$$

in Mod^{E_∞}.

Proof. The proof of the first assertion follows directly from Theorem 3.28 by applying the monoidal functor ι. The proof of the other assertions are the same as the analogous result for CDGA's proved in [GTZ2]. □

3.4. **Higher Hochschild (co)chains models for mapping spaces.** This section is devoted to the relationship in between higher Hochschild chains and mapping spaces. In particular, we prove an E_∞-*algebra version of the Chen iterated integral* morphism studied in [GTZ].

Let A be an E_∞-algebra. Recall that by the coproduct axiom and functoriality of Hochschild chains (see Theorem 3.28, Corollary 3.29), there is a natural equivalence $A \otimes A \cong CH_{S^0}(A)$ of E_∞-algebras as well as a natural E_∞-algebras map $CH_{S^0}(A) \to CH_{pt}(A) \cong A$.

Lemma 3.30. *Let* X, Y *be topological spaces and* $C^*(X)$, $C^*(Y)$ *be their* E_∞-*algebras of cochains. Denote* $\pi_X : X \times Y \to X$ *and* $\pi_Y : X \times Y \to Y$ *the projections onto the first and second factors. The composition,*

$$(13) \quad C^*(X) \otimes C^*(Y) \xrightarrow{\pi_X^* \otimes \pi_Y^*} C^*(X \times Y) \otimes C^*(X \times Y) \xrightarrow{\sim} CH_{S^0}\big(C^*(X \times Y)\big)$$
$$\longrightarrow CH_{pt}\big(C^*(X \times Y)\big) \cong C^*(X \times Y)$$

is a natural morphism of E_∞-algebras. It is further an equivalence under the assumption that $H_*(Y)$ (or $H_*(X)$) is finitely generated in each degree.

Proof. That the maps involved are natural (in $X, Y \in Top_\infty$) maps of E_∞-algebras follows from the functoriality of $X \mapsto C^*(X)$ and the functorial and monoidal properties of the higher Hochschild derived functor (see Theorem 3.28).

We now prove that the map (13) is an equivalence under the assumption that that $H_*(Y)$ is projective, finitely generated in each degree. The idea is to prove that the map (13) is homotopy equivalent to the cross product.

Note that if the ground ring k is a field of characteristic zero, the map (13) induces a map $H^*(X) \otimes H^*(Y) \to H^*(X \times Y)$ which is easily identified with the Künneth morphism since for a graded commutative algebra, the map $A \otimes A \cong CH_{S^0}(A) \to CH_{pt}(A) \cong A$ is given by the multiplication in A (by Corollary 3.7).

For a general ground ring of coefficients, note that as a mere E_1-algebra (via the forgetful functor E_∞-Alg $\hookrightarrow E_1$-Alg), the singular cochain complex $C^*(X)$ is endowed with the (strictly) associative algebra structure given by the cup-product. Let D_+^1, D_-^1 be two open disjoint sub-intervals of D^1 and $i : D_-^1 \coprod D_+^1 \hookrightarrow D^1$ be the inclusion map. By definition (see [L-HA, Lu3, F1]), for any differential graded associative algebra (A, m), the canonical map of *chain complexes* (and not E_1-algebras)

$$A \otimes A \cong \int_{D_-^1 \coprod D_+^1} A \xrightarrow{i_*} \int_{D^1} A \cong A$$

is the multiplication map $m : A \otimes A \to A$ defining the E_1-structure of A. If furthermore (A, m) is actually an E_∞-algebra, by Theorem 3.13 and functoriality of derived Hochschild functor, there is a (homotopy) commutative diagram of chain complexes

and thus, the map (13) is homotopy equivalent, as a map of chain complexes, to

$$(14) \qquad C^*(X) \otimes C^*(Y) \xrightarrow{\pi_X^* \otimes \pi_Y^*} C^*(X \times Y) \otimes C^*(X \times Y) \xrightarrow{\cup} C^*(X \times Y).$$

The cochain complex structure of $C^*(X)$ is the normalization of the cosimplicial k-module $n \mapsto C^n(X)$ so that the above map (14) is the (dual of the) Alexander-Whitney diagonal (in k-Mod$_\infty$):

$$(15) \qquad AW : C^*(X) \otimes C^*(Y) \hookrightarrow \big(C_*(X) \otimes C_*(Y)\big)^\vee \xrightarrow{AW^\vee} C^*(X \times Y).$$

Here the first arrow is the canonical inclusion and the second one the dual of the Alexander-Whitney quasi-isomorphism: $AW : C_*(X \times Y) \xrightarrow{\simeq} C_*(X) \otimes C_*(Y)$. Since

$C_*(X)$, $C_*(Y)$ are complexes of modules and $C_*(Y)$ has finitely generated homology in each degree, both maps in the composition (15) are quasi-isomorphisms; the lemma follows. $\square$

Remark 3.31. The map of E_∞-algebra $C^*(X) \otimes C^*(Y) \to C^*(X \times Y)$ given by Lemma 3.30 is in particular a map of chain complexes. From the last part of the proof of Lemma 3.30, it follows that this map is equivalent in $k\text{-}Mod_\infty$ to the dual of the Alexander-Whitney diagonal (see the maps (14), (15)), *i.e.* the map given by Lemma 3.30 is an E_∞-lifting of the Alexander-Whitney diagonal (also called the cross product).

Let $X_\bullet$ be a simplicial set and Y be a topological space. We define a map

$$ev : Y^{|X_\bullet|} \times \Delta^n \to Y^{X_n}$$

by $ev(f, (t_0, \cdots, t_n)) = g$, where for

$$f : \big(\coprod (X_n \times \Delta^n)/ \sim \big) \to Y \text{ and } (t_0, \cdots, t_n) \in \Delta^n,$$

we have,

$$g(\sigma_n) = f([\sigma_n, (t_0, \cdots, t_n)]), \qquad \text{for } \sigma_n \in X_n.$$

Note that this is a well defined map of cosimplicial topological spaces. In fact, ev is induced by the canonical map $X_n \to Map(\Delta^n, |X_\bullet|)$ given by the unit of the adjunction between simplicial sets and topological spaces.

Applying the E_∞ cochain functor $C^*(-)$ (Example 2.6) yields a natural map

$$(16) \qquad\qquad ev^* : \big(C^*(Y^{X_i})\big)_{(i\in\mathbb{N})} \to \big(C^*(Y^{|X_\bullet|} \times \Delta^i)\big)_{(i\in\mathbb{N})}$$

of simplicial E_∞-algebras.

Lemma 3.32. *The geometric realization of the simplicial E_∞-algebra $\big(C^*(Z \times \Delta^i)\big)_{(i\in\mathbb{N})}$ is naturally equivalent to $C^*(Z)$, as an E_∞-algebra.*

Proof. By Lemma 3.30, there is a natural equivalence $C^*(Z \times \Delta^i) \cong C^*(Z) \otimes C^*(\Delta^i)$ in $E_\infty\text{-}Alg$. This induces an equivalence,

$$C^*(Z) \otimes \big(C^*(\Delta^i)\big)_{(i\in\mathbb{N})} \xrightarrow{\simeq} \big(C^*(Z) \times \Delta^i\big)_{(i\in\mathbb{N})}$$

of simplicial E_∞-algebras. Since the constant map $\Delta^i \to pt$ is a homotopy equivalence, the canonical map $C^*(pt) \to C^*(\Delta^i)$, where $C^*(pt)$ is viewed as a constant simplicial E_∞-algebra, is an equivalence. Composing the above with the equivalence,

$$C^*(Z) \otimes \big(C^*(pt)\big)_{(i\in\mathbb{N})} \xrightarrow{\simeq} C^*(Z) \otimes \big(C^*(\Delta^i)\big)_{(i\in\mathbb{N})}$$

gives rise to an equivalence between $C^*(Z)$ and the constant simplicial E_∞-algebra $C^*(Z \times \Delta^i)$. $\square$

Let $X_\bullet$ be a simplicial set. Iterating Lemma 3.30, we get, for any $n \in \mathbb{N}$, a natural map of E_∞-algebras

$$(17) \qquad\qquad CH_{X_n}(C^*(Y)) \longrightarrow C^*\big(Y^{X_n}\big)$$

Composing the map (17) with the ev^* map in (16), we get a natural morphism of simplicial E_∞-algebras,

$$(18) \qquad \mathcal{I}t : CH_{X_\bullet}^{simp}(C^*(Y)) \longrightarrow C^*\big(Y^{X_\bullet}\big) \xrightarrow{ev^*} C^*\big(Y^{|X_\bullet|} \times \Delta^\bullet\big).$$

The following result is an integral, E_∞-lifting of the iterated integrals [GTZ2].

Theorem 3.33. *The geometric realization of the map*

$$\mathcal{I}t : CH^{simp}_{X_\bullet}(C^*(Y)) \longrightarrow C^*\big(Y^{|X_\bullet|} \times \Delta^\bullet\big)$$

yields a natural (in $X_\bullet$ and Y) morphism of E_∞-algebras

$$\mathcal{I}t : CH_{X_\bullet}(C^*(Y)) \longrightarrow C^*\big(Y^{|X_\bullet|}\big).$$

Further, if $|X_\bullet|$ is n-dimensional (i.e. the highest degree of any non-degenerate simplex is n) and Y is n-connected, then the map $\mathcal{I}t$ is an equivalence.

Proof. Since the natural map (17), $CH^{simp}_{X_\bullet}(C^*(Y)) \to C^*\big(Y^{X_\bullet}\big)$, and the map (16), $C^*\big(Y^{X_\bullet}\big) \overset{ev^*}{\to} C^*\big(Y^{|X_\bullet|} \times \Delta^\bullet\big)$, are simplicial, their realization yields a map of E_∞-algebras

$$CH_{X_\bullet}(C^*(Y)) \to |C^*\big(Y^{|X_\bullet|} \times \Delta^\bullet\big)| \;\cong\; C^*\big(Y^{|X_\bullet|}\big)$$

where the last equivalence follows from Lemma 3.32. This defines the map $\mathcal{I}t$ which is natural by construction.

Now, we assume $|X_\bullet|$ is n-dimensional and Y is n-connected. We only need to check that the underlying map of *cochain complexes* $CH_{X_\bullet}(C^*(Y)) \to C^*\big(Y^{|X_\bullet|} \times \Delta^\bullet\big)$ is an equivalence in the $(\infty, 1)$-category of cochain complexes. The proof of Lemma 3.30 (see Remark (3.31)) implies that the cochain complex morphism

$$CH_{X_\bullet}(C^*(Y)) \longrightarrow C^*(Y^{X_\bullet})$$

is the map induced by the iterated Alexander-Whitney diagonal. Since the geometric realization commutes with the forgetful functor $E_\infty\text{-Alg} \to k\text{-}Mod_\infty$, the geometric realization of the map $C^*(Y^{X_\bullet}) \to C^*\big(Y^{|X_\bullet|} \times \Delta^\bullet\big)$ is equivalent in $k\text{-}Mod_\infty$ to the map induced by the slant products

$$C^\ell(Y^{X_n}) \to C^\ell\big(Y^{|X_\bullet|} \times \Delta^\ell\big) \overset{/[\Delta^n]}{\longrightarrow} C^{\ell-n}\big(Y^{|X_\bullet|}\big)$$

by the fundamental chain $[\Delta^n]$ given by the unique non-degenerate n-simplex of Δ^n.

Hence we have proved that $\mathcal{I}t$ is equivalent in $k\text{-}Mod_\infty$ to the composition

$$\bigoplus_{n \geq 0} CH_{X_n}(C^*(Y)) \longrightarrow \bigoplus_{n \geq 0} C^*(Y^{X_n})$$

$$\overset{\oplus ev^*}{\longrightarrow} \bigoplus_{n \geq 0} C^*\big(Y^{|X_\bullet|} \times \Delta^n\big) \overset{\oplus_{n \geq 0} /[\Delta^n]}{\longrightarrow} C^*\big(Y^{|X_\bullet|}\big).$$

This last map is a quasi-isomorphism under the appropriate assumptions on $X_\bullet$ and Y using the same argument as in [GTZ, PT]. $\qquad\square$

Remark 3.34 (Relationship with Chen integrals)**.** Let $Y = M$ be a manifold and k a field of characteristic zero. Then, by Corollary (3.7) and homotopy invariance of higher Hochschild cochains, there is a natural equivalence of E_∞-algebras

$$CH_{X_\bullet}(C^*(M)) \;\cong\; CH^{cdga}_{X_\bullet}(\Omega(M)).$$

Recall that the slant product is a model for integration over Δ^n. Unfolding the proof of Theorem 3.33 and the construction of the map in Lemma 3.30, one can check that the map $\mathcal{I}t : CH_{X_\bullet}(C^*(M)) \to C^*(M^{|X_\bullet|})$, given by Theorem 3.33, coincides[15]

---

[15]as natural transformations of ∞-functors $sSet_\infty \to E_\infty\text{-Alg}$

with the generalized Chen's iterated integral map defined in [GTZ, Section 2]. In particular, when $X_\bullet$ is the standard simplicial set model of the compact interval or the circle, we recover the original Chen iterated integral construction [Ch]. This justifies our notation $\mathcal{I}t$ for the map defined in Theorem 3.33.

Similarly, the argument of the proofs of Theorem 3.33 and Lemma 3.30 as well as Theorem 3.13 (applied to the forgetful functor from E_∞-algebras to E_1-algebras) show that the iterated integral map $\mathcal{I}t : CH_{X_\bullet}(C^*(Y)) \to C^*\left(Y^{|X_\bullet|}\right)$ given by Theorem 3.33 is homotopy equivalent to the map of differential graded algebras described in [PT]. In particular, for $X = S^1_\bullet$, we recover an E_∞-algebra lift of Jones quasi-isomorphism [Jo].

Similarly, if X is a topological space, by choosing a simplicial model $X_\bullet$ for X (that is a simplicial set with an equivalence $|X_\bullet| \to X$), we get a natural equivalence $CH_X(C^*(Y)) \xrightarrow{\cong} CH_{X_\bullet}(C^*(Y))$ and thus Theorem 3.33 yields the following corollary. Note that an independent proof was obtained by Francis [F2] in the case where X is a manifold.

Corollary 3.35. *The map*

$$\mathcal{I}t : CH_X(C^*(Y)) \xrightarrow{\cong} CH_{X_\bullet}(C^*(Y)) \longrightarrow C^*(Y^X)$$

is a natural (in X, Y) morphism of E_∞-algebras and an equivalence if Y is $\dim(X)$-connected.

We will give a cohomological version of Theorem 3.33. Assume now that X is pointed (by a map $\epsilon : pt \to X$) and choose a pointed simplicial set model $X_\bullet$ of X. By naturality of the map $\mathcal{I}t$ in Theorem 3.33, there is a commutative diagram of E_∞-algebras maps:

$$(19) \qquad \begin{array}{ccc} CH_{X_\bullet}(C^*(Y)) & \xrightarrow{\ \mathcal{I}t\ } & C^*\left(Y^{|X_\bullet|}\right) \\ {\scriptstyle \epsilon_*}\big\uparrow & & {\scriptstyle C^*(\epsilon^*)}\big\uparrow \\ C^*(Y) \cong CH_{pt}(C^*(Y)) & \xrightarrow{\ \mathcal{I}t\ } & C^*\left(Y^{pt}\right) \cong C^*(Y). \end{array}$$

in which the lower map is seen to be the identity map by construction. It follows that $\mathcal{I}t$ is a $C^*(Y)$-E_∞-module map. Denoting $M^\vee = Hom_k(M, k)$ the linear dual of M (equipped with its canonical A-E_∞-structure if M is an A^{op}-E_∞-module), we thus get a map

$$(20) \quad \mathcal{I}t^* : C_*\left(Y^X\right) \cong C_*\left(Y^{|X_\bullet|}\right) \longrightarrow Hom_k\left((C^*(Y^{|X_\bullet|})), k\right)$$
$$\xrightarrow{\cong} Hom_{C^*(Y)}\left((C^*(Y^{|X_\bullet|})), \left(C^*(Y)\right)^\vee\right)$$
$$\xrightarrow{-\circ \mathcal{I}t} Hom_{C^*(Y)}\left(CH_{X_\bullet}(C^*(Y)), \left(C^*(Y)\right)^\vee\right)$$
$$\cong CH^{X_\bullet}\left(C^*(Y), \left(C^*(Y)\right)^\vee\right) \cong CH^X\left(C^*(Y), \left(C^*(Y)\right)^\vee\right)$$

where the first map is biduality morphism, the second map is the canonical isomorphism and the last two isomorphisms are from Definition 3.18.

Corollary 3.36. *The morphism* $\mathcal{I}t^* : C_*\left(Y^X\right) \longrightarrow CH^X\left(C^*(Y), \left(C^*(Y)\right)^{\vee}\right)$ *in* k-Mod_{∞} *is natural in* X *and* Y.

Further, if Y *is* $\dim(X)$-*connected,* X *is compact and the homology groups of* Y *are finitely generated in each degree, then* $\mathcal{I}t^*$ *is a quasi-isomorphism.*

Proof. That $\mathcal{I}t^*$ is natural in X and Y is immediate since all maps involved in its definition are natural in their two arguments.

The assumption Y is $\dim(X)$-connected ensures that $\mathcal{I}t$ is a quasi-isomorphism. Further, for a model $X = |X_\bullet|$ where X_k is finite in every degree, the above assumption together with the assumption on the homology groups of Y ensures that the biduality map $C_*\left(Y^{|X_\bullet|}\right) \longrightarrow Hom_k\left(\left(C^*(Y^{|X_\bullet|})\right), k\right)$ is a quasi-isomorphism as well. Indeed, the connectivity assumption ensures that $H_*\left(Y^{|X_\bullet|}\right)$ is the abutment of the (first quadrant hence) converging spectral sequence given by the simplicial filtration of $X_\bullet$ (and so is $Hom_k\left(\left(C^*(Y^{|X_\bullet|})\right), k\right)$). Its E_1-term is given by the (reduced) homology of $\bigoplus_k C_*(Y^{X_k})$. The finiteness of X_k ensures that each $C_*(Y^{X_k}) \stackrel{\cong}{\Rightarrow} \bigotimes_{X_k} C_*(Y)$ has finite type homology groups in every degree (since Y has), hence is quasi-isomorphic to its bidual from which we deduce that the biduality map is already an isomorphism at the E_1-page. $\qquad\square$

Remark 3.37 (*Weakening the connectivity condition*)**.** The assumption of Y being $\dim(X)$-connected in Theorem 3.33 and Corollary 3.36 is merely there to ensure the convergence of a spectral sequence (introduced in the proof of Corollary 3.36, see [GTZ, PT] for more details), which boils down to the convergence of an Eilenberg-Moore spectral sequence (as explained in [PT, BS]). When $X_\bullet$ is a finite simplicial set, the convergence is ensured under the weaker assumption that Y is connected, nilpotent, with finite homotopy groups in degree less or equal to n as is proved in [F2]. It follows that we have the following proposition.

Proposition 3.38. *Assume* $X \cong |X_\bullet|$ *is compact and* n-*dimensional. Then, Theorem 3.33 and Corollaries 3.35 and 3.36 hold true if* Y *is only connected, nilpotent, with finite homotopy groups in degree less or equal to* n.

4. Algebraic structure of higher Hochschild cochains

4.1. **Wedge and cup products.** Let A be an E_∞-algebra and assume B is an A-algebra, *i.e.*, an E_∞-algebra object in the symmetric monoidal $(\infty, 1)$-category A-Mod^{E_∞} of A-modules, see [L-HA, KM] for details.

Example 4.1. A map $f : A \to B$ of E_∞-algebras induces a natural E_∞-A-algebra structure on B.

Note further that, if B is a unital E_∞-A-algebra, then the map $a \mapsto a \cdot 1_B$ lifts to a map $f : A \to B$ of E_∞-algebras such that the induced E_∞-A-algebra structure on B is equivalent to the original one.

Since there is a canonical map $m_A : A \otimes A \to A$ of E_∞-algebras (Proposition 2.9), any A-module inherits a canonical structure of $A \otimes A$-module (Proposition 2.10).

Lemma 4.2. *Let* $M \in A$-Mod^{E_∞} *be an* A-*module and* X, Y *be pointed topological spaces. There is a natural equivalence*

$$\mu : Hom_{A \otimes A}\left(CH_X(A) \otimes CH_Y(A), M\right) \stackrel{\simeq}{\longrightarrow} CH^{X \vee Y}(A, M)$$

Proof. The excision property yields a natural equivalence

$$CH_{X \vee Y}(A) \cong A \overset{\mathbb{L}}{\underset{A \otimes A}{\otimes}} \Big(CH_X(A) \otimes CH_Y(A) \Big)$$

It follows that we have an equivalence

$$Hom_{A \otimes A} \big(CH_X(A) \otimes CH_Y(A), M \big) \cong Hom_A \big(CH_{X \vee Y}(A), M \big)$$

and the result now follows by Definition 3.16. $\qquad\qquad\square$

Using the above Lemma 4.2, for pointed spaces X, Y and B an A-algebra, we can define the following map

(21)
$$\mu_\vee : CH^X(A, B) \otimes CH^Y(A, B) \longrightarrow Hom_{A \otimes A} \Big(CH_X(A) \otimes CH_Y(A), B \otimes B \Big)$$

$$\overset{(m_B)_*}{\longrightarrow} Hom_{A \otimes A} \Big(CH_X(A) \otimes CH_Y(A), B \Big) \cong CH^{X \vee Y}(A, B)$$

where the first map is given by the tensor products $(f, g) \mapsto f \otimes g$ of functions.

Definition 4.3. We call $\mu_\vee : CH^X(A, B) \otimes CH^Y(A, B) \to CH^{X \vee Y}(A, B)$ the *wedge product* of Hochschild cochains (here we do not require that B is unital).

Note that this construction was already studied in some particular cases in our previous papers [G1, GTZ].

Example 4.4 (Small model for CDGA's). If A, B are actually CDGA's and given finite pointed set models $X_\bullet, Y_\bullet$ of X, Y, the map $\mu_\vee$ can be combinatorially described as follows. We have two cosimplicial chain complexes $CH^{X_\bullet}(A, B) \otimes CH^{Y_\bullet}(A, B)$ (with the diagonal cosimplicial structure) and $CH^{X_\bullet \vee Y_\bullet}(A, B)$. There is a cosimplicial map $\tilde{\mu} : CH^{X_\bullet}(A, B) \otimes CH^{Y_\bullet}(A, B) \to CH^{X_\bullet \vee Y_\bullet}(A, B)$ given, for any $f \in CH^{X_n}(A, B) \cong Hom_A(A^{\otimes \#X_n}, B)$, $g \in CH^{X_n}(A, B) \cong Hom_A(A^{\otimes \#Y_n}, B))$ by

$$\mu(f, g)(a_0, a_2, \ldots a_{\#X_n}, b_2, \ldots, b_{\#Y_n}) = \pm a_0.f(1, a_2, \ldots a_{\#X_n}).g(1, b_2, \ldots, b_{\#Y_n})$$

where a_0 corresponds to the element indexed by the base point of $X_n \vee Y_n$ (the sign is given by the usual Koszul-Quillen sign convention). Composing the map $\tilde{\mu}$ with the dual of the Eilenberg-Zilber quasi-isomorphism realizes the wedge map (21):

$$\mu_\vee : CH^X(A, B) \otimes CH^Y(A, B) \to CH^{X \vee Y}(A, B).$$

Proposition 4.5. *The map $\mu_\vee$ is associative,* i.e.*, there is a commutative diagram*

$$
\begin{array}{ccc}
CH^X(A, B) \otimes CH^Y(A, B) \otimes CH^Z(A, B) & \overset{\mu_\vee \otimes id}{\longrightarrow} & CH^{X \vee Y}(A, B) \otimes CH^Z(A, B) \\
{\scriptstyle id \otimes \mu_\vee} \downarrow & & \downarrow {\scriptstyle \mu_\vee} \\
CH^X(A, B) \otimes CH^{Y \vee Z}(A, B) & \overset{\mu_\vee}{\longrightarrow} & CH^{X \vee Y \vee Z}(A, B)
\end{array}
$$

in $k\text{-}Mod_\infty$.

Proof. It follows from the associativity of the wedge product of spaces and tensor products of E_∞-algebras as used in Lemma 4.2 and Proposition 2.9. $\qquad\square$

Let X be a homotopy coassociative co-H-space, *i.e.*, a topological space X endowed with a continuous map $\delta_X : X \to X \vee X$ which is co-associative (up to homotopy). Note that all suspension spaces has this structure, even though they are rarely manifolds. Then, by functoriality, we get a morphism $\delta_X^* : CH^{X \vee X}(A, B) \to CH^X(A, B)$.

Corollary 4.6. *Assume X is a homotopy coassociative co-H-space. The composition*

$$\cup_X : \quad CH^X(A, B) \otimes CH^X(A, B) \xrightarrow{\mu_\vee} CH^{X \vee X}(A, B) \xrightarrow{\delta_X^*} CH^X(A, B),$$

called the cup-product, induces a structure of graded associative algebra on the cohomology groups $HH^X(A, B)$. It is further unital if B is unital and X counital.

Proof. The associativity follows from Proposition 4.5 and Proposition 3.22. When B has an unit 1_B and X is counital, then it follows from the contravariance of Hochschild cochains with respect to maps of pointed spaces that the unit of $\cup_X$ is given by the canonical map

(22) $$k \xrightarrow{1_B} B \cong CH^{pt}(A, B) \xrightarrow{(X \to pt)^*} CH^X(A, B).$$

Indeed, the two compositions $\left(id \vee (X \to pt) \right) \circ \delta_X$ and $\left((X \to pt) \vee id \right) \circ \delta_X$ are homotopical to the identity. Further, the composition

$$CH^X(A, B) \otimes k \xrightarrow{id \otimes 1_B} CH^X(A, B) \otimes CH^{pt}(A, B) \xrightarrow{\mu_\vee} CH^X(A, B)$$

is the identity map of $CH^X(A, B)$ (which can be checked on any simplicial set model of X). $\qquad\square$

In particular, the pinching map $S^d \to S^d \vee S^d$ obtained by collapsing the equator to a point induces a cup product $\cup_{S^d} : CH^{S^d}(A, B) \otimes CH^{S^d}(A, B) \to CH^{S^d}(A, B)$ for Hochschild cohomology over spheres for any E_∞-algebra A and A-algebra B. For CDGA's, this cup-product agrees by definition and Remark 3.24 with the one introduced by the first author in [G1].

Example 4.7 (Cup-product on the standard simplicial model for spheres). In the case of spheres and *CDGA's*, there is an explicit description of the cup product if one uses the standard model of the dimension d sphere. Recall that the standard simplicial set model of the circle S^1 is the simplicial set, denoted $(S_{st}^1)_\bullet$, generated by a unique non-degenerate simplex of dimension 1. Thus $(S_{st}^1)_n := n_+$ where $n_+ = \{0, \cdots, n\}$ has $\{0\}$ for its base point see [G1, GTZ, P]. The standard simplicial set $(S_{st}^d)_\bullet$ is the iterated smash product $(S_{st}^d)_\bullet = (S_{st}^1)_\bullet \wedge \cdots \wedge (S_{st}^1)_\bullet$ so that $(S_{st}^d)_n = (n^d)_+$. Using this standard simplicial set model, we have an equivalence

$$CH^{S^d}(A, M) \cong CH^{(S_{st}^d)_\bullet}(A, M) \cong Hom_k\left(A^{\otimes(\bullet)^d}, M \right)$$

see [G1] (in particular, for the description of the differential on the right hand side). Note that we do not know any simplicial map $(S_{st}^d)_\bullet \to (S_{st}^d)_\bullet \vee (S_{st}^d)_\bullet$ modeling the pinching map. However, there is a simplicial map $q : sd_2((S_{st}^d)_\bullet) \to (S_{st}^d)_\bullet \vee (S_{st}^d)_\bullet$ modeling it. Here $sd_2((S_{st}^d)_\bullet)$ is the edgewise subdivision [McC] (also see [GTZ, § 3.3.2] for examples of applications in the context of higher Hochschild complexes) of $(S_{st}^d)_\bullet$; it can be seen as the simplicial model of the circle obtained by gluing two intervals at their endpoints. In other words $sd_2((S_{st}^d)_n := (2n+1)_+ = \{0, \ldots, 2n+1\}$

pointed in 0. The map $q : sd_2((S^d_{st})_n) \to (S^d_{st})_n \vee (S^d_{st})_n \cong \{1, \ldots, n\} \cup \{0\} \cup \{n + 2, \ldots 2n+1\}$ identifies $n+1$ with 0. The cup-product is thus realized by the induced map

$$\left(CH^{(S^d_{st})\bullet}(A,B)\right)^{\otimes 2} \xrightarrow{\mu_\vee} CH^{(S^d_{st})\bullet \vee (S^d_{st})\bullet}(A,B) \xrightarrow{q^*} CH^{sd_2((S^d_{st})\bullet)}(A,B).$$

There is also a cochain complex map (*not* induced by a map simplicial sets) making $CH^{(S^d_{st})\bullet}(A,B)$ a differential graded associative algebra on the nose described in [G1]. Let $f \in C^{(S^d_{st})_p}(A,B) \cong Hom_k(A^{\otimes(p^d)}, B)$ and $g \in C^{(S^d_{st})_q}(A,B) \cong Hom_k(A^{\otimes(q^d)}, B)$. Define $f \cup_0 g \in C^{(S^d_{st})_{p+q}}(A,B) \cong Hom_k(A^{\otimes((p+q)^d)}, B)$ by

$$(23) \quad f \cup_0 g\left((a_{i_1,\ldots,i_d})_{1 \leq i_1,\ldots,i_d \leq p+q}\right)$$

$$= f((a_{i_1,\ldots,i_d})_{1 \leq i_1,\ldots,i_d \leq p}) g((a_{i_1,\ldots,i_d})_{p+1 \leq i_1,\ldots,i_d \leq p+q}) \prod a_{j_1,\ldots,j_d}$$

where the last product is over all indices which are not in the argument of f or g. Note that for $d = 1$, this is the formula of the usual cup-product for Hochschild cochains as in [Ge] and for $n = 2$, this is the Riemann sphere product as defined in [GTZ].

The following lemma is proved using a straightforward computation

Lemma 4.8. *Let A be a CDGA and B a commutative differential graded A-algebra. Then $\left(CH^{(S^d_{st})\bullet}(A,B), d, \cup_0\right)$ (where d is the total differential as in [G1, GTZ]) is an associative differential graded algebra (and unital if B is unital).*

The above explicit formula for the cup-product realizes the cup-product induced by the co-H space structures of the spheres.

Proposition 4.9. *The natural equivalence $CH^{S^d_{st}\bullet}(A,B) \cong CH^{S^d}(A,B)$ is an equivalence of E_1-algebras (with E_1-structures induced by Lemma 4.8 and Corollary 4.6).*

Proof. The proof in the case $d = 2$ is given in the proof of [GTZ, Proposition 3.3.17]. The argument for general d is the same. $\square$

We finish this section by giving a more structured version of the wedge product. The wedge product (21) is the degree 0-component of a higher homotopical tower of wedge products. Since B is an E_∞-algebra, it is in particular, by restriction of structure, an E_n-algebra for any positive integer n.

Assume, that A and B are algebras over the (chains on the) linear isometry operad ([KM]) viewed as algebras over $C_*(\mathcal{C}_n)$, the (chains on the) little dimension n-cubes operad. For any $c \in C_*(\mathcal{C}_n(r))$, we have a map $m_B(c) : B^{\otimes r} \longrightarrow B$. Similarly to the wedge product, we can thus define the composition

$$(24) \quad \mu_\vee(c) : \bigotimes_{i=1}^r CH^{X_i}(A,B) \longrightarrow Hom_{A^{\otimes r}}\left(\bigotimes_{i=1}^r CH_{X_i}(A), B^{\otimes r}\right)$$

$$\xrightarrow{(m_B(c))_*} Hom_{A^{\otimes r}}\left(\bigotimes_{i=1}^r CH_{X_i}(A), B\right) \cong CH^{\vee_{i=1}^r X_i}(A,B).$$

where the first map is given by the tensor products of morphisms.

Remark 4.10. When B is a CDGA, then all operations $\mu_\vee(c)$ vanishes if c is not of degree 0.

Example 4.11 (Strict chain model for algebras over an E_∞ Hopf-operad)**.** The map $\mu_\vee(c)$ can be defined similarly at the chain level whenever A, B are algebras over an E_∞-operad $(\mathcal{E}(n))_n$ which is further an Hopf-operad, that is, is equiped with a diagonal of operads $\mathcal{E}(n) \to \mathcal{E}(n) \otimes \mathcal{E}(n))$. In that case one gets map $\mu_\vee(c)$ for any $c \in \mathcal{E}(n)$. A nice model of such an E_∞ Hopf operad is the Barratt-Eccles operad [BF].

4.2. **A natural E_d-algebra structure on Hochschild cochains modeled on d-dimensional spheres.** We have already seen the definition of the cup product for Hochschild cochains modeled on spheres for E_∞-algebras, see Corollary 4.6. We now turn to the full E_d-structure on $CH^{S^d}(A, B)$. In [G1], the first author proved that if A is a CDGA and B is a commutative A-algebra (for example $B = A$), there is a natural E_n-algebra structure on $CH^{S^n}(A, B)$. In this section, we recall this construction in the context of ∞-categories of E_∞-algebras. We will relate this construction to centralizers in the sense of Lurie [L-HA, Lu3] in Section 6.

Recall that we denote $\mathcal{C}_d$ the usual d-dimensional little cubes operad (as an operad of topological spaces) whose associated ∞-operad is a model for $\mathbb{E}_d^\otimes$, see[L-HA, Lu3]. $\mathcal{C}_d(r)$ is the configuration space of r many d-dimensional open cubes in I^d. Any element $c \in \mathcal{C}_d(r)$ defines a map $pinch_c : S^d \to \bigvee_{i=1\dots r} S^d$ by collapsing the complement of the interiors of the r cubes to the base point. The maps $pinch_c$ assemble together to give a continuous map

$$(25) \qquad pinch : \mathcal{C}_d(r) \times S^d \longrightarrow \bigvee_{i=1\dots r} S^d.$$

Note that the map $pinch$ preserves the base point of S^d, hence passes to the pointed category.

For any topological space X, the singular set functor $X \mapsto \Delta_\bullet(X) := Map(\Delta^\bullet, X)$ defines a (fibrant) simplicial set model of X. Hence, applying the singular set functor to the above map $pinch$, the contravariance[16] of Hochschild cochains (see Proposition 3.9 and Proposition 3.22) and the wedge product (24) $\mu_\vee$, we get, for all $r \geq 1$, a morphism

$$(26) \quad pinch^*_{S^d,r} : C_*\big(\mathcal{C}_d(r)\big) \otimes \Big(CH^{S^d}(A, B)\Big)^{\otimes r}$$

$$\stackrel{diag \otimes id}{\longrightarrow} C_*\big(\mathcal{C}_d(r)\big)^{\otimes 2} \otimes \Big(CH^{S^d}(A, B)\Big)^{\otimes k}$$

$$\stackrel{\mu_\vee(diag^{(2)})_*}{\longrightarrow} C_*\big(\mathcal{C}_d(r)\big) \otimes CH^{\bigvee_{i=1}^r S^d}(A, B)$$

$$\stackrel{pinch^*}{\longrightarrow} CH^{S^d}(A, B)$$

in $k\text{-}Mod_\infty$. Here $diag : C_*\big(\mathcal{C}_d(r)\big) \to C_*\big((\mathcal{C}_d(r))^2\big) \stackrel{AW}{\to} \Big(C_*\big(\mathcal{C}_d(r)\big)\Big)^{\otimes 2}$ is the diagonal and $diag^{(1)}$, $diag^{(2)}$ its components.

Theorem 4.12. *Let A be an E_∞-algebra and B an E_∞-A-algebra (not necesssarily unital). The collection of maps $(pinch_{S^d,k})_{k\geq 1}$ makes $CH^{S^d}(A, B)$ an E_d-algebra*

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[16]with respect to maps of topological spaces

(naturally in A, B), which is unital if B is unital. Further, the underlying E_1-structure of $CH^{S^d}(A, B)$ agrees with the one given by Corollary 4.6.

Note that the last map $C_*\big(\mathcal{C}_d(r)\big) \otimes CH^{\bigvee_{i=1}^r S^d}(A, B) \xrightarrow{pinch^*} CH^{S^d}(A, B)$ in the definition of the composition (26) is just the map dual to the one associated to

$$\widetilde{pinch} : \mathcal{C}_d(r) \longrightarrow Map_{E_\infty\text{-Alg}}(CH_{S^d}(A), CH_{\bigvee_{i=1}^r S^d}(A)))$$

in Remark 3.10 (see formula (9)).

Proof. To prove the first statement we need to prove that the morphisms $pinch^*_{S^d, r}$ are compatible with the operadic composition in $C_*\big(\mathcal{C}_d(r)\big)$, the singular chains on the little d-dimensional cubes. Since the diagonal $diag : C_*\big(\mathcal{C}_d(r)\big) \to \Big(C_*\big(\mathcal{C}_d(r)\big)\Big)^{\otimes 2}$ is a map of ∞-operads, by Proposition 4.5 and Proposition 3.11 (as in Example 3.12), the statement reduces to the commutativity of the following diagram for every $j \in \{1, \ldots, k\}$

$$
\begin{array}{ccc}
\mathcal{C}_d(k) \times \mathcal{C}_d(\ell) \times S^d & \xrightarrow{pinch} & \mathcal{C}_d(\ell) \times \bigvee_{i=1\ldots k} S^d \\
\downarrow{\circ_j \times id_{S^d}} & & \downarrow{id_{\bigvee_{i=1\ldots j-1} S^d} \times pinch \times id_{\bigvee_{i=j+1\ldots k} S^d}} \\
\mathcal{C}_d(k+\ell) \times S^d & \xrightarrow{pinch} & \bigvee_{i=1\ldots k+\ell} S^d.
\end{array}
$$

In other words, it reduces to the fact that the pointed sphere S^d is a $\mathcal{C}_d$-coalgebra in the category of pointed topological spaces endowed with the monoidal structure given by the wedge product.

The underlying E_1-structure is given by any element in $\mathcal{C}_d(2)$ generating the homology group $H_0(\mathcal{C}_d(2), \mathbb{Z}) \cong \mathbb{Z}$. We can, for instance, take the configuration of the two open cubes $(-1, 0)^d$ and $(0, 1)^d$ in $(-1, 1)^d$. It follows immediately with this choice, that the associated E_1-structure is given by the cup-product $\cup_{S^d}$ of Corollary 4.6 up to equivalences of E_1-algebras. The unit is given by the map (22) as in Corollary 4.6.

$\square$

This theorem will be generalized in Theorem 7.10 below to also include generalized sphere topology operations. The naturality in A and B means that if C is a B-E_∞-algebra map, then, there is an E_d-algebra homomorphism

$$CH^{S^d}(A, B) \otimes CH^{S^d}(B, C) \longrightarrow CH^{S^d}(A, C)$$

see Proposition 6.15 and Theorem 6.8.

Remark 4.13. For $d > 1$, Theorem 4.12 implies that the cup-product makes the Hochschild cohomology groups $HH^{S^d}(A, B)$ a graded *commutative* algebra (and not only associative as in the case $d = 1$).

Further, when $B = A$ (endowed with its canonical A-algebra structure), the E_d-structure can actually be lifted naturally to an E_{d+1}-structure; see Theorem 6.28.(3).

Remark 4.14. Similarly to Example 4.7, it is possible (but a bit tedious) to give explicit description of the higher $\cup_i$-products on the standard models of the spheres. Details are left to the interested reader.

The core of the proof of Theorem 4.12 is the E_d-co-H-space structure of the sphere. We say that a pointed topological space X is an E_d-co-H-space if it is an E_d-coalgebra in the category of pointed spaces with monoidal structure given by the wedge product.

In other words, there are continuous maps $\mathcal{C}_d(k) \times X \to \bigvee_{i=1\ldots k} X$ which are compatible with the operadic composition in $\mathcal{C}_d$. Mimicking the proof of Theorem 4.12 gives the following enhancement of Corollary 4.6:

Corollary 4.15. *Let X be an E_d-co-H-space, A an E_∞-algebra and B an E_∞-A-algebra. Then there is a natural (in X in E_d-co-H-space, A and B) E_d-algebra structure on $CH^X(A, B)$ refining the cup-product of Corollary 4.6*

Example 4.16 (Smooth CDGA). In characteristic zero, there is an equivalence $E_n\text{-Alg} \cong H_*(\mathcal{C}_n)\text{-Alg}$ between the ∞-categories of E_n-algebras and (homotopy) $H_*(\mathcal{C}_n)$-algebras induced by any choice of formality of the little n-disks-operad. Note that for $n = 1$ the latter operad $H_*(\mathcal{C}_1)$ is the operad of associative algebras while for $n \geq 2$, $H_*(\mathcal{C}_n)$ is the operad governing P_n-algebras.

The next proposition shows that for *free* graded commutative algebras, the homotopy P_n-structure given by Theorem 4.12 is trivial.

Here a P_n-algebra stands for a differential graded commutative unital algebra $(B, d, \cdot)$ equipped with a (homological) degree $n-1$ bracket which makes the iterated suspension $A[1-n]$ a differential graded Lie algebra. The bracket and product are further required to satisfy the graded Leibniz identity, see paragraph 6.5.1 below.

If P is a P_n-algebra and C a P_n-coalgebra, we can form the convolution P_n-algebra $Hom(C, P)$ (as in [Ta2]).

Proposition 4.17. *Let $A = (\mathrm{Sym}(V), d)$ and $B = (\mathrm{Sym}(W), b)$ be differential free graded commutative algebras and assume $n \geq 2$.*

- *There is an natural quasi-isomorphism*

$$CH_{S^n}(A) \xrightarrow{\simeq} (\mathrm{Sym}(V \otimes H_\bullet(S^n)), \partial) \cong (\mathrm{Sym}(V \oplus V[-n], \partial)$$

  of CDGA. Here the right hand side is equipped with the unique differential such that for any $v \in V$, $\partial(v) = d(v)$ and $\partial v[-n] = (-1)^n s_n(d(v))$ where s_n is the unique derivation satisfying $s_n(w) = w[-n]$, $s_n(w[-n]) = 0$ for $w \in V$.

- *There is an natural equivalence of (homotopy) P_n-algebras*

$$CH^{S^n}(A, B) \cong Hom_{\mathrm{Sym}(V)}\Big(\mathrm{Sym}(V \otimes H_\bullet(S^n)), \mathrm{Sym}(W)\Big)$$

  where the right hand side is endowed with the convolution P_n-algebra structure[17] given by the linear isomorphism

$$Hom_{\mathrm{Sym}(V)}\Big(\mathrm{Sym}(V \otimes H_\bullet(S^n)), \mathrm{Sym}(W)\Big) \cong Hom(\mathrm{Sym}(V[-n]), \mathrm{Sym}(W)\Big).$$

  where $\mathrm{Sym}(V[-n])$ is the cofree coproartinian graded cocommutative coalgebra seen as a P_n-coalgebra with trivial bracket.

Proof. The first claim is (a special case of) the Hochschild-Kostant-Rosenberg Theorem for higher Hoschild homology proved in [P] and Remark 3.3. Note that the quasi-isomorphism is obtained by the degeneration of a spectral sequence which is

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[17]which has a zero bracket

natural in both A and maps of topological spaces [P, §2]. In fact, in our case we can use Proposition 3.9: one has natural equivalences $CH_{S^n}(A) \cong C_*(S^n) \overset{\mathbb{L}}{\underset{\mathbb{E}_\infty^\otimes}{\otimes}} A$ and

$$(\mathrm{Sym}(V \otimes H_\bullet(S^n)), \partial) \cong H_*(S^n) \overset{\mathbb{L}}{\underset{\mathbb{E}_\infty^\otimes}{\otimes}} A.$$

The equivalence is then induced by the fact that S^n is formal (this is essentially the approach in [P]); alternatively, one can use Corollary 3.15 if V is negatively graded.

The first claim thus also implies that

$$CH^{S^n}(A, B) \cong Hom_{\mathrm{Sym}(V)}\Big(\mathrm{Sym}(V \otimes H_*(S^n)), \mathrm{Sym}(W)\Big)$$

as cochain complexes and that this equivalence is an equivalence of (homotopy) P_n-algebras, where the P_n-structures are given as algebras over the operad $\big(H_*(\mathcal{C}_n(r))\big)$. The right hand side is equipped with a $\big(H_*(\mathcal{C}_n(r))\big)$-algebra structure given by the action of $\big(H_*(\mathcal{C}_n(r))\big)$ on $H_*(S^n)$ and thus on $H_*(S^n) \overset{\mathbb{L}}{\underset{\mathbb{E}_\infty^\otimes}{\otimes}} A$; this action being similar to the one on $CH_{S^n}(A) \cong C_*(S^n) \overset{\mathbb{L}}{\underset{\mathbb{E}_\infty^\otimes}{\otimes}} A$ given by Theorem 4.12. Namely it is given by the composition:

$$(27) \quad pinch^*_{S^d,r} : H_*\big(\mathcal{C}_n(r)\big) \otimes \Big(Hom_{\mathrm{Sym}(V)}\big(\mathrm{Sym}(V \otimes H_*(S^n)), \mathrm{Sym}(W)\big)\Big)^{\otimes r}$$

$$\overset{\mu_B}{\longrightarrow} H_*\big(\mathcal{C}_n(r)\big) \otimes Hom_{\mathrm{Sym}(V)}\Big(\mathrm{Sym}\big(V \otimes H_*\big(\bigvee_{i=1}^r S^n\big)\big), \mathrm{Sym}(W)\Big)$$

$$\overset{pinch^*}{\longrightarrow} Hom_{\mathrm{Sym}(V)}\Big(\mathrm{Sym}(V \otimes H_*(S^n)), \mathrm{Sym}(W)\Big).$$

Here μ_B stands for the multiplication in $B = \mathrm{Sym}(W)$. For degree reason, this $H_*\big(\mathcal{C}_n(r)\big)$-algebra structure is the one of a CDGA endowed with zero bracket. In particular it corresponds to the convolution P_n-algebra mentioned in the Proposition. $\square$

5. Factorization homology and E_n-modules

In this section, for $n = \{1, 2, \ldots, \} \cup \{+\infty\}$, we collect some results on the category of E_n-modules over an E_n-algebra A. In particular we identify it with the category of left modules over the factorization homology $\int_{S^{n-1}} A$ in § 5.2. Then we apply this to E_∞-modules to show the existence and uniqueness of the lift of Poincaré duality in the category of E_∞-modules in § 5.4. These results are latter used in § 6 and § 7.

We first start by presenting a Factorization algebra point of view on E_n-modules.

5.1. Stratified factorization algebras and E_n-modules.
One can define a notion of locally constant factorization algebra for stratified manifolds as well as factorization homology for such spaces. We refer to [AFT] and [G2] for details.

In this paper, we will essentially only need very special and easy cases: the disk and the sphere with a marked point.

Let X be a Hausdorff paracompact topological space. By a *stratification of X*, we mean an union of a sequence of closed subspaces $\emptyset = X_{-1} \subset X_0 \subset X_1 \subset$

$\cdots \subset X_n = X$ such that, for any point $x_i \in X_i$, there is a neighborhood U_{x_i} and a filtration preserving homeomorphism $U_{x_i} \overset{\phi}{\simeq} \mathbb{R}^i \times C(L)$ in X where $C(L)$ is the (open) cone on a stratified space of dimension $n - i - 1$.

Note that $X_{i+1} \setminus X_i$ is not necessarily connected nor non-empty. In all the examples considered in this paper, it will nevertheless be a smooth manifold of dimension $i + 1$. We call the connected components of $X_i \setminus X_{i-1}$ the strata of dimension i of X.

We define the **index** of an open subset $U \subset X$ to be the *smallest integer j* such that $U \cap X_j \neq \emptyset$.

Definition 5.1. An an open subset U of X is called a (stratified) *disk* if there is a filtration preserving homeomorphism $U \cong R^i \times C(L)$ with L stratified of dimension $n - i - 1$, and i is the index of U.

A *factorization algebra* $\mathcal{F}$ over a *stratified manifold* $\emptyset \subset X_0 \subset X_1 \subset \cdots \subset X_n = X$ is called *locally constant* if the following condition is satisfied:

if $U \hookrightarrow V$ is an inclusion of (stratified) disks of *same index*[18] and further V intersects only one stratum of $X_i \setminus X_{i-1}$ where i is the index of V, then we require that the structure map $\mathcal{F}(U) \to \mathcal{F}(V)$ is a quasi-isomorphism.

We will also say that $\mathcal{F}$ is *stratified locally constant* when we want to insist on the stratification.

Example 5.2 (Pointed disk). We write D_*^n for the pointed Euclidean open disk viewed as a stratified manifold. It has only two strata: a dimension 0 stratum given by its center and the dimension n-stratum given by the complement of the center. In other words $D_0^n = \{0\} = D_1^n \cdots = D_{n-1}^n$ and $D_n^n = \mathbb{R}^n$.

Thus a factorization algebra (or an $N(\text{Disk}(D^n)$-algebra) on D_*^n is locally constant if the structure map $\mathcal{F}(U) \to \mathcal{F}(V)$ is an equivalence when $U \subset V$ are open disks such that either U contains the base point or V is included in the n-stratum $D^n - \{0\}$. In other words, we do *not* require $\mathcal{F}(U) \to \mathcal{F}(V)$ to be an equivalence if V contains the base point while U does not.

We let $Fac_{D_*^n}^{lc}$ be $((\infty, 1))$-category of stratified locally constant factorization algebras on the pointed disk.

Definition 5.3.   • We denote $Fac_{D_*^n}^{lc, res}$, the $(\infty, 1)$-category

$$Fac_{D_*^n}^{lc, res} := \text{Fac}_{D_*^n}^{lc} \times_{\text{Fac}_{\mathbb{R}^n \setminus \{0\}}^{lc}} \text{Fac}_{\mathbb{R}^n}^{lc}$$

of pairs $(\mathcal{M}, \mathcal{A})$ of locally constant factorization algebras on respectively D_*^n and D^n together with an equivalence of factorization algebras $\mathcal{M}_{|D^n - \{0\}} \overset{\simeq}{\Rightarrow} \mathcal{A}_{|D^n - \{0\}}$ between the restrictions of $\mathcal{M}$ and $\mathcal{A}$ to $D^n - \{0\}$.

• Fix $\mathcal{B} \in \text{Fac}_{D^n}^{lc}$. We denote $Fac_{D_*^n|\mathcal{B}}^{lc}$ the $(\infty, 1)$-category

$$Fac_{D_*^n|\mathcal{B}}^{lc} := \text{Fac}_{D^n}^{lc} \times_{\text{Fac}_{\mathbb{R}^n \setminus \{0\}}^{lc}} \{\mathcal{B}\},$$

that is the $(\infty, 1)$-category of pairs $(\mathcal{M}, \mathcal{A})$ as above such that $\mathcal{A}$ is (equipped with an equivalence with) $\mathcal{B}$.

---

[18]that is, for all $j = 0, \ldots, n-1$, either $V \cap X_j = \emptyset$ or both $U \cap X_j$ and $V \cap X_j$ are non empty

Remark 5.4. Given an object $(\mathcal{M}, \mathcal{A}) \in Fac_{D^n_*}^{lc,res}$, the factorization algebra $\mathcal{A}$ is essentially determined by $\mathcal{M}$, since, by the locally constant condition, it is essentially defined by its restriction to any open ball in D^n, thus to any open ball included in $D^n - \{0\}$.

The same is *locally* true for any object $\mathcal{N}$ of $Fac_{D^n_*}^{lc}$. Indeed, restricting to any disk D of $D^n - \{0\}$ yields a locally constant factorization algebra on the disk and thus an E_n-algebra $A_D \cong \mathcal{N}(D)$. For any two disks D_1, D_2, the E_n-algebra A_{D_1} and A_{D_2} are equivalent, but such an equivalence depends on a choice of a bigger disk containing both of them. Thus, the main difference between $Fac_{D^n_*}^{lc,res}$ and $Fac_{D^n_*}^{lc}$ is that we assume that these equivalences can be made canonically, which amout to the fact that $\mathcal{N}_{|D \setminus \{0\}}$ is equivalent ot the restriction of a fixed factorization algebra on $\mathbb{R}^n$.

Example 5.5. For $n = 1$, the category $Fac_{D^1_*}^{lc,res}$ is equivalent to the category of all bimodules over an E_1-algebra. However, $Fac_{D^1_*}^{lc}$ is equivalent to the category of all bimodules, that is the category whose objects are (A, B)-bimodules for some E_1-algebras A, B which may be non-equivalent.

Theorem 2.29 has an analogue for modules:

Proposition 5.6 ([G2])**.**
- *There is an equivalence between the $(\infty, 1)$-categories Mod^{E_n} of all E_n-modules (§ 2.2) and $Fac_{D^n_*}^{lc,res}$, the locally constant factorization algebras on the pointed disk as in Definition 5.3.*
- *Let A be an E_n-algebra corresponding to a factorization algebra $\mathcal{A} \in Fac_{\mathbb{R}^n}^{lc}$ under Theorem 2.29. Then the above equivalence restricts to an equivalence*

$$A\text{-}Mod^{E_n} \cong Fac_{D^n_* | \mathcal{A}}^{lc}.$$

Note that the pushforward $D^n_* \to pt$ realizes the forgetful functor $Mod^{E_n}(\mathcal{C}) \to \mathcal{C}$ of § 2.2. Further, as noted in Remark 5.4, fixing any Euclidean sub-disk $D \subset D^n_* - \{0\}$ we get a functor $Fac_{D^n_*}^{lc,res} \to Fac_D^{lc}$ which is equivalent to the functor $\iota : Mod^{E_n} \to E_n$-Alg, *i.e.*, to the forgetful functor $(\mathcal{M}, \mathcal{A}) \mapsto \mathcal{A}$.

Forgetting the stratification yields a canonical functor $Fac_{D^n}^{lc} \to Fac_{D^n_*}^{lc,res}$ realizing the canonical functor E_n-Alg $\to Mod^{E_n}$ (which views an E_n-algebra as a module over itself in a canonical way).

Remark 5.7 (Induced E_n-module structure associated to an E_n-algebra map)**.** Let A be an E_n-algebra and $f : A \to B$ an E_n-algebra map and let B be endowed with the induced A-E_n-module structure. This module structure has an easy description in terms of factorization algebras. Denote $\mathcal{A} : U \mapsto \int_U A$ and $\mathcal{B} : V \mapsto \int_V B$ be the associated factorization algebras on $\mathbb{R}^n$ (see Theorem 2.20). By Proposition 5.6 and Proposition 2.25, the data of the A-E_n-module structure on B is equivalent to the data of a parametrized factorization algebra. Thus, to any embedding $\coprod_{i=0}^r \phi_i : \coprod_{i=0}^r \mathbb{R}^n \to \mathbb{R}^n$ (with $\phi_0(0) = 0$) and commutative diagram

$$
\begin{array}{ccc}
\coprod_{i=0}^r \mathbb{R}^n & \xrightarrow{\quad h \quad} & \mathbb{R}^n \\
{\scriptstyle \coprod_{i=0}^r \phi_i} \searrow & & \swarrow {\scriptstyle \psi} \\
& \mathbb{R}^n &
\end{array}
$$

of embeddings, one can associate a natural[19] map

$$(28) \qquad \mathcal{A}(\phi_1(\mathbb{R}^n)) \otimes \cdots \otimes \mathcal{A}(\phi_r(\mathbb{R}^n)) \otimes \mathcal{B}(\phi_0(\mathbb{R}^n)) \longrightarrow \mathcal{B}(\psi(\mathbb{R}^n)).$$

This map (28) is very simple to describe, it is the composition

$$\int_{\phi_1(\mathbb{R}^n)} A \otimes \cdots \otimes \int_{\phi_r(\mathbb{R}^n)} A \otimes \int_{\phi_0(\mathbb{R}^n)} B$$

$$\xrightarrow{\left(\otimes_{i=1}^r \int_{\phi_i(\mathbb{R}^n)} f \right) \otimes id} \int_{\phi_1(\mathbb{R}^n)} B \otimes \cdots \otimes \int_{\phi_r(\mathbb{R}^n)} B \otimes \int_{\phi_0(\mathbb{R}^n)} B$$

$$\longrightarrow \int_{\psi(\mathbb{R}^n)} B \cong \mathcal{B}(\psi(\mathbb{R}^n)).$$

where the last map is given by the factorization algebra structure of $\mathcal{B}$, *i.e.*, the E_n-algebra structure of B.

Now, let $g : B \to C$ be another E_n-algebra map endowing C with an A-E_n-module structure; let $\mathcal{C} : U \mapsto \int_U C$ be the associated factorization algebra. Then a map A-E_n-modules $h : B \to C$ is equivalent to the data of a stratified parametrized factorization algebra map $\int_U h : \mathcal{B}(U) \cong \int_U B \to \int_U C \cong \mathcal{C}(U)$ such that, for all $\phi_0, \ldots, \phi_r$ and ψ as above, the following diagram

$$
\begin{array}{ccc}
\left(\otimes_{i=1}^r \int_{\phi_i(\mathbb{R}^n)} A \right) \otimes \int_{\phi_0(\mathbb{R}^n)} B & \xrightarrow{\left(\otimes \int_{\phi_i(\mathbb{R}^n)} f \right) \otimes id} \otimes_{i=0}^r \int_{\phi_i(\mathbb{R}^n)} B \longrightarrow \int_{\psi(\mathbb{R}^n)} B \\
\left(\otimes id \right) \otimes \int_{\phi_0(R^n)} h \downarrow & \qquad\qquad\qquad\qquad \downarrow \int_{\psi(\mathbb{R}^n)} h \\
\left(\otimes_{i=1}^r \int_{\phi_i(\mathbb{R}^n)} A \right) \otimes \int_{\phi_0(\mathbb{R}^n)} C \xrightarrow[\left(\otimes \int_{\phi_i(\mathbb{R}^n)} g \circ f \right) \otimes id]{} \otimes_{i=0}^r \int_{\phi_i(\mathbb{R}^n)} C \longrightarrow \int_{\psi(\mathbb{R}^n)} C
\end{array}
$$

is commutative.

5.2. **Universal enveloping algebra of an E_n-algebra.** In this section, we will recall some general results that are needed, among other places, in the proof of Proposition 6.2. We start with the following very useful result describing the universal enveloping algebra of an E_n-algebra in terms of factorization homology. Note that universal enveloping algebras of E_n-algebras are given by the left adjoint of the forgetful functor E_n-Alg $\to k$-Mod$_\infty$ (for instance see [F1]).

Proposition 5.8 (Francis, Lurie)**.** *Let A be an E_n-algebra $(n \in \mathbb{N})$. The category A-Mod^{E_n} is equivalent as a symmetric monoidal $(\infty, 1)$-category to the category of left modules over the factorization homology $\int_{S^{n-1}}(A)$, with respect to the canonical outward n-framing on $S^{n-1} \subset \mathbb{R}^n$.*

Proof. This is proved in [F1] and can also be found in [Lu3, L-HA]. Note that by the ∞-version of the Barr-Beck theorem [Lu1, L-HA] for any E_n-algebra A, there is an E_n-enveloping algebra $U_A^{(n)} \in E_1$-Alg with a natural equivalence $U_A^{(n)}$-LMod $\cong$ A-Mod^{E_n}, see *loc. cit* and also [Fre]. Now the result follows from the natural equivalence $U_A^{(n)} \xrightarrow{\simeq} \int_{S^{n-1}} A$ see [F1, Proposition 3.19]. $\square$

---

[19]with respect to composition of embeddings, that is satisfies the usual associativity condition of the structure maps of a prefactorization algebra in the sense of [CG]

This Lemma extends to the case $n = \infty$, see Lemma 5.15 and more importantly Theorem 5.13 below.

Remark 5.9. In terms of factorization algebra, the equivalence in Proposition 5.8 can be thought of as the pushforward of factorization algebras. A Euclidean norm of a vector defines a canonical map $N : D_*^n \to [0,1)_*$, where $[0,1)_*$ is the half open interval with a unique closed stratum given by the point 0. The $(\infty, 1)$-category of locally constant factorization algebra on the stratified manifold $[0,1)_*$ is equivalent to the $(\infty, 1)$-category $LMod$. The equivalence of Proposition 5.8 is then just induced by the pushforward $N_* : Fac_{D_*^n}^{lc} \to Fac_{[0,1)_*}^{lc}$ by N. See [G2] for details.

We will later need the following lemma, which expresses the compatibility of the equivalence of categories given by Proposition 5.8 with the inclusions of E_{n+1}-algebras inside E_n-algebras. We feel this lemma is of independent interest anyhow. Suppose X is a codimension 1 submanifold of an n-framed manifold and Y endowed with a trivialization $\psi : X \times \mathbb{R} \hookrightarrow Y$ of a tubular neighborhood in Y. Then, for any E_n-algebra A, there is a canonical map $\psi : \int_X A \to \int_Y A$ (which depends on the trivialization).

Lemma 5.10. *Let A be an E_{n+1}-algebra and $\phi_n : S^{n-1} \times \mathbb{R} \hookrightarrow S^n$ the inclusion of an open (tubular) neighborhood of the equatorial sphere $S^{n-1} = S^n \cap (\mathbb{R}^n \times \{0\})$ inside S^n. The following diagram, in which the vertical arrows are given by Proposition 5.8, is commutative,*

$$
\begin{array}{ccc}
A\text{-}Mod^{E_{n+1}} & \longrightarrow & A\text{-}Mod^{E_n} \\
\Big\downarrow{\simeq} & & \Big\downarrow{\simeq} \\
(\int_{S^n} A)\text{-}LMod & \xrightarrow{\phi_n^*} & (\int_{S^{n-1}} A)\text{-}LMod
\end{array} \quad .
$$

Proof. The universal property of the E_n-enveloping algebra $U_A^{(n)}$ implies that the map of ∞-operad $\mathbb{E}_n^\otimes \to \mathbb{E}_{n+1}^\otimes$ (see § 2.2) yields a canonical map of E_1-algebras $U_A^{(n)} \to U_A^{(n+1)}$. It remains to identify the composition $\theta_n : \int_{S^{n-1}} A \cong U_A^{(n)} \to U_A^{(n+1)} \cong \int_{S^n} A$ with ϕ_n to prove the lemma. From the proof of [F1, Proposition 3.19], we know that $U_A^{(n)}$ is computed by the colimit of a (simplicial) diagram,

$$
(29) \qquad \coprod_{K \in Fin} \mathbb{E}_n^\otimes\left(K \coprod \{pt\}\right) \otimes A^{\otimes K} \Longleftarrow \coprod_{\mathbb{E}_n^\otimes(J,I)} \mathbb{E}_n^\otimes\left(I \coprod \{pt\}\right) \otimes A^{\otimes J} \cdots .
$$

Similarly, $\int_{S^{n-1}} A$ can be computed as the colimit of a similar diagram,

$$
(30)
$$
$$
\coprod_{K \in Fin} Emb^{fr}\left(\coprod_K D^n, S^{n-1} \times \mathbb{R}\right) \otimes A^{\otimes K} \Longleftarrow \coprod_{\mathbb{E}_n^\otimes(J,I)} Emb^{fr}\left(\coprod_I D^n, S^{n-1} \times \mathbb{R}\right) \otimes A^{\otimes J} \cdots ,
$$

where Emb^{fr} denotes the space of framed embeddings.

Furthermore, the equivalence $U_A^{(n)} \xrightarrow{\cong} \int_{S^{n-1}} A$ is induced by the canonical maps $\mathbb{E}_n^\otimes\left(K \coprod \{pt\}\right) \to Emb^{fr}\left(\coprod_K D^n, S^{n-1} \times \mathbb{R}\right)$ obtained by translating the disk labeled by the distinguished point to the origin; see the proof of [F1, Proposition 3.19].

The natural map $U_A^{(n)} \to U_A^{(n+1)}$ is induced by the natural maps $\mathbb{E}_n^{\otimes}\left(K \coprod \{pt\}\right) \to$ $\mathbb{E}_{n+1}^{\otimes}\left(K \coprod \{pt\}\right)$ in diagram (29). Since the natural map of ∞-operads $\mathbb{E}_n^{\otimes} \to \mathbb{E}_{n+1}^{\otimes}$ is given by sending n-dimensional disks D to $D \times \mathbb{R}$, we get commutative diagrams

$$
\begin{array}{ccc}
\mathbb{E}_n^{\otimes}\left(K \coprod \{pt\}\right) & \longrightarrow & \mathbb{E}_{n+1}^{\otimes}\left(K \coprod \{pt\}\right) \\
\downarrow & & \downarrow \\
Emb^{fr}\left(\coprod_K D^n, S^{n-1} \times \mathbb{R}\right) & \longrightarrow & Emb^{fr}\left(\coprod_K D^{n+1}, S^n \times \mathbb{R}\right)
\end{array}
$$

where the lower map is induced by the embedding

$$\phi_n \times \mathbb{R} : (S^{n-1} \times \mathbb{R}) \times \mathbb{R} \hookrightarrow S^n \times \mathbb{R}$$

prescribed in the assumptions of the lemma. It follows that $\theta_n : \int_{S^{n-1}} A \cong U_A^{(n)} \to U_A^{(n+1)} \cong \int_{S^n} A$ is obtained by taking the colimit of these lower maps

$$Emb^{fr}\left(\coprod_K D^n, S^{n-1} \times \mathbb{R}\right) \xrightarrow{\phi_n \times \mathbb{R}} Emb^{fr}\left(\coprod_K D^{n+1}, S^n \times \mathbb{R}\right)$$

applied to diagram (30), which, by definition, is the map $\phi_n : \int_{S^{n-1}} A \to \int_{S^n} A$. □

Remark 5.11 (The trivial E_n-A-module structure on A). It follows from the axioms of factorization homology (see [Lu3, L-HA, F1] or [GTZ2, Section 6]) that for any E_n-algebra A, there is a natural equivalence $A \cong \int_{D^n} A$ in $k - Mod_\infty$. Note that D^n has an immediate trivialization $S^{n-1} \times D^1 \cong D^n \setminus \{0\}$ of a complement of a point (and in fact of any complement of a closed disk). Hence, there is a natural left $(\int_{S^{n-1}} A)$-module structure on $\int_{D^n} A$ see [Lu3, L-HA], [F1, Section 3] or [GTZ2, Section 6.3] for details. Note that this left $\left(\int_{S^{n-1}} A\right)$-module structure is given by a map

$$\int_{S^{n-1}} A \otimes \int_{D^n} A \cong \int_{\left(S^{n-1} \times D^1\right) \coprod D^n} A \longrightarrow \int_{D^n} A$$

induced by any embedding $\left(S^{n-1} \times D^1\right) \coprod D^n \hookrightarrow D^n$ mapping D^n onto a subdisk $D(0,r) \subset D^n$ (for some radius $r > 0$) and $S^{n-1} \times D^1$ onto a sub-annulus included in $D^n \setminus D(0,r)$.

By Proposition 5.8, we get a natural A-E_n-module structure on A which relates to the canonical A-E_n-module structure of A as follows.

Lemma 5.12. *The natural equivalence $A \cong \int_{D^n} A$ is an equivalence of A-E_n-modules.*

Proof. Consider a framed embedding of $D^n \hookrightarrow \mathbb{R}^n$. Since $D^n \setminus \{0\}$ is framed, the result follows from [F1, Remark 3.26]. In fact, the proof of [F1, Proposition 3.19] applied to A and not the unit object of $\mathcal{C} = k\text{-}Mod_\infty$ gives an equivalence of left $\int_{S^{n-1}} A$-modules between A viewed as an $(\int_{S^{n-1}} A)$-module and $\int_{D^n} A$. □

5.3. **Application of higher Hochschild chains to prove Theorem 5.13.** For E_∞-algebras, Proposition 5.8 has a simpler and well-known (see [L-HA, KM, Fre]) form, see Theorem 5.13 below. In this section, we recall this result and then give an independent proof using the formalism of factorization homology/higher Hochschild chains.

The following theorem is due to Lurie [L-HA, Proposition 4.4.1.4], [Lu2] and also appeared independently in the work of Fresse [Fre].

Theorem 5.13. *Let A be an E_∞-algebra. There is an equivalence of symmetric monoidal ∞-categories between the category A-Mod^{E_∞} of E_∞ A-Modules and the category of left A-modules (where A is viewed as an E_1-algebra). In particular:*

- *Any left A-module can be promoted into an E_∞-A-module (up to quasi-isomorphisms)*
- *Any map $f : M \to N$ of left A-modules can be lifted to a map of E_∞-modules (up to a contractible family of choices)*

The theorem allows us to reduce the study of E_∞-modules on $C^*(X)$ to the study of left modules on the (differential graded) associative algebra $(C^*(X), \cup)$, for instance see § 5.4. Also, see Example 5.18 and Remark 5.19 for a more explicit description of the lifts of left modules into E_∞-ones.

Remark 5.14. When A is an E_∞-algebra the categories of left and right modules over A (viewed as an E_1-algebra) are equivalent. Hence, one can replace left modules by right modules in Theorem 5.13.

The rest of this section is devoted to an alternative proof of Theorem 5.13 using § 5.2 and higher Hochschild theory. We first start with the following analogue of Proposition 5.8.

Lemma 5.15. *Let A be an E_∞-algebra. The category A-Mod^{E_∞} of E_∞-A-Modules is equivalent as a symmetric monoidal $(\infty, 1)$-category to the category of left modules over the derived Hochschild chains $CH_{S^\infty}(A)$, viewed as an E_1-algebra by forgetting extra structure.*

Proof. By Theorem 3.13, there is a canonical equivalence $\int_{S^n} A \cong CH_{S^n}(A)$ for any $n \in \mathbb{N}$.

The maps of operad $\mathbb{E}_i^\otimes \to \mathbb{E}_{i+1}^\otimes$ are induced by the maps $\mathbb{R}^i \cong \mathbb{R}^i \times \{0\} \hookrightarrow \mathbb{R}^{i+1}$ which, by restriction induces canonical maps $S^{i-1} \cong S^i \cap (\mathbb{R}^i \times \{0\}) \hookrightarrow S^i$, and, by functoriality, maps $\phi_i : CH_{S^{i-1}}(A) \to CH_{S^i}(A)$.

By Lemma 5.10 (and Theorem 3.13), we get a commutative diagram

$$\ldots \overset{\subset}{\longrightarrow} A\text{-}Mod^{E_{n+1}} \overset{\subset}{\longrightarrow} A\text{-}Mod^{E_n} \overset{\subset}{\longrightarrow} \ldots \overset{\subset}{\longrightarrow} A\text{-}Mod^{E_1} \qquad .$$
$$\Big\downarrow \simeq \qquad\qquad \Big\downarrow \simeq \qquad\qquad\qquad \Big\downarrow \simeq$$
$$\ldots \longrightarrow CH_{S^n}(A)\text{-}LMod \overset{\phi_n^*}{\longrightarrow} CH_{S^{n-1}}(A)\text{-}LMod \longrightarrow \ldots \overset{\phi_0^*}{\longrightarrow} CH_{S^0}(A)\text{-}LMod$$

From Lemma 5.16, we deduce a natural equivalence

$$A\text{-}Mod^{E_\infty} \cong \lim_{n \geq 1} CH_{S^n}(A)\text{-}LMod.$$

Mimicking the proof of [GTZ2, Lemma 5.1.3], we get a natural equivalence

$$\varinjlim \Big(CH_{S^0}(A) \to CH_{S^1}(A) \to \cdots \to CH_{S^n}(A) \to \ldots \Big) \overset{\simeq}{\longrightarrow} CH_{S^\infty}(A).$$

It follows that we have an equivalence $CH_{S^\infty}(A)\text{-}LMod \to \lim_{n \geq 1} CH_{S^n}(A)\text{-}LMod$ and the lemma follows. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Lemma 5.16. *Let A be an E_∞-algebra, then $CH_{S^\infty}(A)$ is canonically equivalent to A as an E_∞-algebra. In particular, there is a canonical equivalence*

$$A\text{-}Mod^{E_n} \cong CH_{S^\infty}(A) - Mod^{E_n}$$

for any $n \in \{0, 1, \ldots, \infty\}$.

Proof. This follows from Theorem 3.28 since S^∞ has a deformation retraction to a point. $\square$

The canonical map $\mathbb{E}_{n-1}^\otimes \to \mathbb{E}_n^\otimes$ yields a natural functor $A\text{-}Mod^{E_n} \to A\text{-}Mod^{E_{n-1}}$ for any E_n-algebra A.

Lemma 5.17. *Let A be an E_∞-algebra. Then $A\text{-}Mod^{E_\infty}$ is the (homotopy) limit*

$$A\text{-}Mod^{E_\infty} \cong lim\left(\cdots \to A\text{-}Mod^{E_n} \to A\text{-}Mod^{E_{n-1}} \to \cdots \to A\text{-}Mod^{E_1}\right).$$

Proof. Recall that $\mathbb{E}_\infty^\otimes \cong \underset{n\geq 1}{colim}\, \mathbb{E}_n^\otimes$ [Lu3, L-HA]. Since we have commuting restriction maps $A\text{-}Mod^{E_\infty} \to A\text{-}Mod^{E_n}$ ($n \in \mathbb{N}$), there is a canonical map

$$\tau : A\text{-}Mod^{E_\infty} \longrightarrow \underset{n\geq 1}{lim}\, A\text{-}Mod^{E_n}.$$

We want to prove that this map τ is an equivalence. Given any E_n-algebra A and an E_n-A-module M, the trivial extension $A \oplus M$ has a natural structure of E_n-algebra. The trivial extension functor $M \mapsto A \oplus M$ is a (natural in A) equivalence of ∞-categories between $A\text{-}Mod^{E_n}$ and $E_n\text{-}\mathrm{Alg}_{/A}$ which, by naturality, commutes with the restriction of structure functors $A\text{-}Mod^{E_n} \to A\text{-}Mod^{E_{n-1}}$ and $E_n\text{-}\mathrm{Alg}_{/A} \to E_{n-1}\text{-}\mathrm{Alg}_{/A}$. It follows that any object of $\underset{n\geq 1}{lim}\, A\text{-}Mod^{E_n}$ is equivalent to an object of $\underset{n\geq 1}{lim}\, E_n\text{-}\mathrm{Alg}_{/A}$. Such an object is a (homotopy type of) chain complex equipped with compatible E_n-structures for all $n \geq 1$, thus is an E_∞-algebra. It is also endowed with compatible augmentations of E_n-algebras to A. Hence we get a map

$$\varphi : \underset{n\geq 1}{lim}\, E_n\text{-}\mathrm{Alg}_{/A} \longrightarrow E_\infty\text{-}\mathrm{Alg}_{/A}$$

which is a quasi-inverse of the canonical map

$$\tau : E_\infty\text{-}\mathrm{Alg}_{/A} \longrightarrow \underset{n\geq 1}{lim}\, E_n\text{-}\mathrm{Alg}_{/A}$$

induced by the restrictions functors. The result now follows by applying the (quasi-inverse of the) trivial extension functor. $\square$

Proof of Theorem 5.13. The first statement follows from Lemmas 5.15 and 5.16 and the last two statements are consequences of the first one. $\square$

Example 5.18. Let A be an E_∞-algebra and M be a left $CH_{S^\infty}(A)$-module (here $CH_{S^\infty}(A)$ is equipped with its canonical E_1-structure by restriction of structure along the operad maps (2)). Since $CH_{S^\infty}(A)$ is in fact an E_∞-algebra, for any $n \in \{1, \ldots, \infty\}$, it is canonically equivalent to its opposite E_n-algebra $CH_{S^\infty}(A)^{op}$. The equivalence is explicitly given by the antipodal map $S^\infty \overset{\mathrm{ant}}{\to} S^\infty$ (and functoriality of Hochschild chains). Thus, there is a canonical structure of $CH_{S^\infty}(A) \otimes CH_{S^\infty}(A)$-$E_\infty$-modules on $CH_{S^\infty}(A)$. By restriction of structure, the map

$$CH_{S^\infty}(A) \overset{1\otimes id}{\longrightarrow} CH_{S^\infty}(A) \otimes CH_{S^\infty}(A)$$

endows $CH_{S^\infty}(A)$ with a right module structure over itself (viewed as an E_1-algebra) which commutes with the $CH_{S^\infty}(A)$-module structure induced by the map

$$CH_{S^\infty}(A) \xrightarrow{id\otimes 1} CH_{S^\infty}(A) \otimes CH_{S^\infty}(A).$$

This extra structure of $CH_{S^\infty}(A)$ endows the tensor product (of a right and left module over $CH_{S^\infty}(A)$ viewed as an E_1-algebra)

$$CH_{S^\infty}(A) \otimes_{CH_{S^\infty}(A)} M$$

with a structure of E_∞-module.

Remark 5.19 (Iterative liftings). One can lift any left A-module to an A-E_∞-module in the same way as in Example 5.18.

By restriction of structure, any left A-module map between E_∞-A-modules can be lifted to a map of E_n-modules (for $n \in \mathbb{N} \cup \{\infty\}$). For the sake of explicit computations, we now explain how to realize this concretely using the higher Hochschild functor. Let M, N be E_∞-modules over A. By restriction of structure we get in particular left A-modules structure on M and N. Let $f : M \to N$ be a map of left A-modules.

The natural structure of $A \otimes A^{op}$-E_∞-module structure on A also yields, by restriction of structure, Proposition 5.8 and Lemma 3.13, a natural structure of left $A \otimes \left(CH_{S^{n-1}}(A)\right)^{op}$-module on A where the left factor A is viewed as an E_1-algebra only.

It follows that, viewing N as left A-module only by restriction, $Hom_A(A, N)$ is endowed with a natural left $CH_{S^{n-1}}(A)$-module structure and further that we have a natural isomorphism of left $CH_{S^{n-1}}(A)$-modules $Hom_A(A, N) \xrightarrow{\simeq} N$ (given by $f \mapsto f(1)$). We get similarly a left $CH_{S^{n-1}}(A) \otimes A^{op}$-module structure on A and a natural equivalence of left $CH_{S^{n-1}}(A)$-modules $A \otimes_A M \xrightarrow{\simeq} M$ (where the tensor product is over A viewed as an E_1-algebra only).

We now explain how to lift f to an E_n-module map (here $n \in \{1, \dots, \infty\}$). The canonical map $D^n \to pt$ being a homotopy equivalence, we get a natural quasi-isomorphism $CH_{D^n}(A) \xrightarrow{\simeq} A$ with quasi-inverse induced by the map sending a point to the center of D^n. The canonical map $S^{n-1} \hookrightarrow D^n$ given by the boundary of D^n gives a map of E_∞-algebra $CH_{S^{n-1}}(A) \to CH_{D^n}(A)$ which, together with the previous morphism, endow $CH_{D^n}(A)$ with a structure of left $CH_{S^{n-1}}(A) \otimes A^{op}$-module. We thus have a natural quasi-isomorphism (of chain complexes)

$$Hom_{CH_{S^{n-1}}(A)}(M, N) \cong Hom_{CH_{S^{n-1}}(A)}\big(A \otimes_A M, Hom_A(A, N)\big)$$
$$\xrightarrow{\sim} Hom_{CH_{S^{n-1}}(A)}\big(CH_{D^n}(A) \otimes_A M, Hom_A(A, N)\big)$$
$$\xrightarrow{\simeq} Hom_A\big(A \otimes_{CH_{S^{n-1}}(A)} CH_{D^n}(A) \otimes_A M, N)\big)$$

where the last map is the canonical isomorphism

$$\psi \mapsto \Big(x \otimes_{CH_{S^{n-1}}(A)} y \otimes_A m \mapsto \pm\psi(y \otimes_A m)(x)\Big)$$

where the sign $\pm$ is given by the Koszul-Quillen signs rule.

Note that there is an equivalence of E_∞-algebras $A \otimes_{CH_{S^{n-1}}(A)} CH_{D^n}(A) \xleftarrow{\simeq} CH_{S^n} A$ which induces, by restriction, a quasi-isomorphism of left $A \otimes A^{op}$-modules

(induced by the choice of two antipodal points on S^n). We thus get a quasi-isomorphism

$$Hom_A\big(A \otimes_{CH_{S^{n-1}}(A)} CH_{D^n}(A) \otimes_A M, N)\big) \xrightarrow{\simeq} Hom_A\big(CH_{S^n}(A) \otimes_A M, N)\big)$$

hence an explicit quasi-isomorphism

(31) $$Hom_{CH_{S^{n-1}}(A)}(M, N) \xrightarrow{\simeq} Hom_A\big(CH_{S^n}(A) \otimes_A M, N\big).$$

The canonical map $S^n \to pt$ also yield a map of E_∞-algebras $CH_{S^n}(A) \to A$, which, by restriction of structures is also a map of left $A \otimes A^{op}$-modules. Hence; we have a natural morphism

(32) $$Hom_A(M, N) \cong Hom_A(A \otimes_A M, N) \longrightarrow Hom_A\big(CH_{S^n}(A) \otimes_A M, N\big).$$

Thus, for any n, we can lift the left module map $f \in Hom_A(M, N)$ to a map of left $CH_{S^{n-1}}(A)$-module hence a map of A-E_n-module (by Proposition 5.8 or Lemma 5.15). Note that by Lemma 5.16 the map $CH_{S^\infty}(A) \to A$ is a quasi-isomorphism, hence the map (32) is a quasi-isomorphism for $n = \infty$ and the lift of f is unique in that case. However, lift of f to E_n-module maps are not unique in general for finite n.

Remark 5.20 (CDGA case)**.** When A is a CDGA (over a field), the Hochschild chain complex $CH_{D^n}(A)$ is a semi-free module over $CH_{S^{n-1}}(A)$ (provided we choose a simplicial model $D_\bullet^n$ for D^n and take $\partial D_\bullet^n$ as a model for S^{n-1}), and therefore all equivalences involved in the maps (31) and (32) can be (quasi-)inverted by standard homological algebra techniques. Note that when $A = C^*(X)$ is the algebra of cochains for a topological space X, the map of E_∞-algebras $CH_{S^n}(A) \to A$ can be factorized as a map

$$CH_{S^n}(C^*(X)) \to C^*(Map(S^n, X)) \to C^*(X)$$

where the last map is induced by the map $X \to Map(S^n, X)$ that sends every point in $p \in X$ to a constant map $C_p : S^n \to X$ defined as $C_p(a) = p$. Hence, in the special case $n = 1$, we recover the construction of [FTV], which was done for $M = C^*(X)$ and $N = C_*(X)$ only.

5.4. **Poincaré duality as a map of E_∞-modules.** We apply the results of the previous sections to achieve an E_∞-lift of the Poincaré duality isomorphism for a closed manifold.

Let C be an E_∞-coalgebra and let $C^\vee = Hom_k(C, k)$ be its linear dual endowed with its canonical E_∞-algebra structure; in particular, $C^\vee$ is naturally a E_∞-$C^\vee$-module. Similarly, the dual space $(C^\vee)^\vee$ is E_∞-$C^\vee$-module. Note that $C \subset (C^\vee)^\vee$ has an induced E_∞-$C^\vee$-module structure. If C is an E_1-coalgebra, then $C^\vee$ is an E_1-algebra has well.

We recall the following standard definition of the cap-product

Definition 5.21. Let C be an E_1-coalgebra. The *cap-product* is the composition

$$\cap : C^\vee \otimes C \xrightarrow{id \otimes \Delta} C^\vee \otimes C \otimes C \xrightarrow{\langle -, - \rangle \otimes id} C$$

where $\Delta : C \to C \otimes C$ is the coproduct (given by the E_1-structure of C) and $\langle -, - \rangle : C^\vee \otimes C \to k$ is the duality pairing. The cap-product of $x \in C^\vee$, $y \in C$ will be denoted $x \cap y$ as usual.

The cap-product map $\cap : C^\vee \otimes C \to C$ allows us to associate to any cycle c in C, a map of *left $C^\vee$-modules* $\cap c : C^\vee \to C$, $x \mapsto x \cap c$, called *the cap-product by c.* Note that this construction only uses the underlying E_1-coalgebra structure of C (even if C is an E_∞-algebra).

Corollary 5.22. *Let C be an E_∞-coalgebra. The cap product by c, $C^\vee \xrightarrow{\cap c} C$, lifts uniquely to a map of E_∞-modules $\rho_c : C^\vee \to C$ which is an equivalence if $\cap c$ is a quasi-isomorphism.*

Proof. The cap-product by c, denoted $\cap c : C^\vee \to C$, is a map of left modules over $C^\vee$ (seen as an E_1-algebra) because $\Delta : C \to C \otimes C$ is an E_1-coalgebra structure. It follows from Theorem 5.13 that the unique lift exists. If $\cap c$ is a quasi-isomorphism, then it is an invertible element in $Hom_{C^\vee}(C^\vee, C)$ and thus its lift is invertible in $Hom_{CH_{S^\infty}(C^\vee)}(C^\vee, C)$ (see Remark 5.19 for an explicit description of the equivalence). □

We now specialize to the case where C is the singular cochain of a space. Let us recall the following definition.

Definition 5.23. By a *Poincaré duality space*, we mean a topological space X together with a choice of cycle $[X] \in C_d(X)$ (for some integer d) such that that cap-product $C^*(X) \xrightarrow{\cap [X]} C_{d-*}(X)$ by $[X]$ is a quasi-isomorphism. The integer d is called the *dimension* of X and denoted $d = \dim(X)$.

Example 5.24. An oriented[20] closed manifold M of dimension $\dim(M)$ (in the usual manifold sense of dimension) is a Poincaré duality space of dimension $\dim(M)$.

Remark 5.25. By definition, the cap product by a class $[X]$ is given by $f \mapsto \sum f([X]^{(1)}) [X]^{(2)}$ (where we denote $\Delta([X]) := \sum [X]^{(1)} \otimes [X]^{(2)}$ the coproduct). It follows that the image $\chi_X(H^*(X))$ is a finitely generated sub k-module of $H_*(X)$. Thus, *if X is a Poincaré duality space, its (co)homology groups are finitely generated* (as k-modules).

Let X be a Poincaré duality space (for instance, an oriented closed manifold) with fundamental class $[X]$. Recall that $C_*(X)$ is the singular cochains of X with its natural structure of E_∞-coalgebra (Example 2.6). Its linear dual $C^*(X)$ is endowed with the dual E_∞-algebra structure. Then, by Corollary 5.22 we have

Corollary 5.26. *Let $(X, [X])$ be a Poincaré duality space. The cap-product by $[X]$ induces a quasi-isomorphism of E_∞-$C^*(X)$-modules*

$$(33) \qquad\qquad \chi_X : C^*(X) \xrightarrow{\simeq} C_*(X)[\dim(X)]$$

realizing the (unique) E_∞-lift of the Poincaré duality isomorphism.

In other words, a Poincaré duality space X (in the sense of Definition 5.23) gives rise to a canonical equivalence of E_∞-modules between its singular chains and cochains.

Definition 5.27. Let $(X, [X])$, $(Y, [Y])$ be Poincaré duality space (of same dimension $d = \dim(X) = \dim(Y)$). A map of Poincaré duality space $f : (X, [X]) \to$

---

[20]with respect to the homology with coefficients in the ground ring k

$(Y, [Y])$ is a map of topological spaces $f : X \to Y$ such that the following diagram is commutative

$$
\begin{array}{ccc}
C^*(X) & \xrightarrow{\cap [X]} & C_*(X)[d] \\
f^* \uparrow & & \downarrow f_* \\
C^*(Y) & \xrightarrow{\cap [Y]} & C_*(Y)[d]
\end{array}
$$

in $C^*(Y)$-Mod^{E_∞}.

Example 5.28. Let $f : M \to N$ be a continuous map between oriented smooth manifolds such that $f_*([M]) = [N]$. Then f induces a map of Poincaré duality spaces.

6. Centralizers

Given any map $f : A \to B$ of E_∞-algebras, by Theorem 4.12, there is a natural E_n-algebra structure on $CH^{S^n}(A, B)$. On the other hand, for a map $f : A \to B$ of E_n-algebras, Lurie [L-HA, Lu3] constructs an E_n-algebra $\mathfrak{z}(f)$. We prove in § 6.3 that $CH^{S^n}(A, B)$ is equivalent to $\mathfrak{z}(f)$ as an E_n-algebra. This will be a corollary of a more general construction for E_n-Hochschild cohomology. Indeed, Hochschild cochains modeled on spheres $CHS^n(A, B)$ is a special case of E_n-Hochschild cohomology $HH_{\mathcal{E}_n}(A, B)$ of A, B viewed as E_n-algebras, see § 6.1. In Section 6.2, in the general case of a map $f : A \to B$ between E_n-algebras, we will give an explicit E_n-algebra structure on $HH_{\mathcal{E}_n}(A, B)$, similar to the one obtained in Section 4.2. We then prove that $HH_{\mathcal{E}_n}(A, B)$ is equivalent to $\mathfrak{z}(f)$. We will apply these results to the case $A = B$, *i.e.*, to get solutions of the (higher) Deligne conjecture in § 6.4.

6.1. E_n-Hochschild cohomology and Hochschild cohomology over S^n. There is an (operadic) notion of cohomology for E_n-algebras closely related to their deformation complexes, see [F1, KS]. We start with the following definition.

Definition 6.1. Let M be an E_n-A-module over an E_n-algebra A. The E_n-Hochschild complex of A with values M, denoted by $HH_{\mathcal{E}_n}(A, M)$, is by definition (see [F1]) $RHom_A^{\mathcal{E}_n}(A, M)$. Here $RHom_A^{\mathcal{E}_n}$ denotes the hom space in the (∞-)category A-Mod^{E_n} of E_n-A-modules.

In particular, if A is an E_m-algebra with $m \in \{n, n+1 \ldots, \infty\}$ (for instance a CDGA), we can define the E_n-Hochschild complex of A $HH_{\mathcal{E}_n}(A, A)$.

In the case where A is an E_∞-algebra, its E_n-Hochschild complex can be described by higher Hochschild cochains over the n-dimensional sphere S^n:

Proposition 6.2. *If A is an E_∞-algebra and M an E_∞-A-module, there is a natural equivalence*

$$
HH_{\mathcal{E}_n}(A, M) \cong CH^{S^n}(A, M),
$$

where CH^{S^n} denotes the derived higher Hochschild cochain functor.

Proof. Given left modules M, N over an E_1-algebra R, we write $RHom_R^{left}(M, N)$ for the hom space in the (∞,1)-category R-$LMod$ of left R-modules. By Proposition 5.8, there is an equivalence of ∞-categories A-$Mod^{E_n} \cong \left(\int_{S^{n-1}} A \right)$-$LMod$ where $\int_{S^{n-1}} A$ is the factorization homology of S^{n-1} with value in A. Here, S^{n-1} is endowed with the n-framing induced by the natural embedding $S^{n-1} \hookrightarrow \mathbb{R}^n$. Thus we have a sequence of natural equivalences

$$
\begin{aligned}
HH_{\mathcal{E}_n}(A, M) \ &\cong \ RHom_A^{\mathcal{E}_n}(A, M) \\
&\cong \ RHom^{left}_{\int_{S^{n-1}} A}(A, M) \\
&\cong \ RHom^{left}_{\int_{S^{n-1}} A}\left(\int_{D^n} A, M \right) \\
&\cong \ RHom^{left}_{CH_{S^{n-1}}(A)}\left(CH_{D^n}(A), M \right) \\
&\cong \ RHom_A^{left}\left(CH_{D^n}(A) \otimes^{\mathbb{L}}_{CH_{S^{n-1}}(A)} A, M \right) \\
&\cong \ RHom_A^{left}\left(CH_{S^n}(A), M \right) \\
&\cong \ CH^{S^n}(A, M).
\end{aligned}
$$

Here we are using the natural equivalence of E_n-A-modules $\int_{D^n} A \xrightarrow{\simeq} A$ (Lemma 5.12).

Note that, by Theorem 3.13, when A is further an E_∞-algebra, we get a natural equivalence of E_1-algebras $\int_{S^{n-1}}(A) \cong CH_{S^{n-1}}(A)$ and by Theorem 3.28 a natural equivalence of E_∞-algebras $CH_{S^n}(A) \cong CH_{D^n}(A) \otimes^{\mathbb{L}}_{CH_{S^{n-1}}(A)} A$. $\qquad\square$

Remark 6.3. Let A, B be E_n-algebras and $f : A \to B$ an E_n-algebra map so that B inherits an A-E_n-module structure. By Definition 6.1, Proposition 5.8 and Lemma 5.12, we have natural equivalences

$$
HH_{\mathcal{E}_n}(A, B) \ \cong \ RHom_A^{\mathcal{E}_n}\left(A, B \right) \ \cong \ RHom^{left}_{\int_{S^{n-1}} A}\left(\int_{D^n} A, \int_{D^n} B \right).
$$

6.2. **The E_n-algebra structure on $\mathcal{E}_n$-Hochschild cohomology $HH_{\mathcal{E}_n}(\mathbf{A},\mathbf{B})$.** In this section, we construct an explicit E_n-algebra structure on the $\mathcal{E}_n$-Hochschild cohomology $HH_{\mathcal{E}_n}(A, B)$ of an E_n-algebra A with value in an E_n-algebra B endowed with an A-E_n-module structure given by a map $A \to B$ of E_n-algebras.

We fix a map $f : A \to B$ of E_n-algebras and we endow B with the induced A-E_n-module structure so that we have E_n-Hochschild cohomology[21] $HH_{\mathcal{E}_n}(A, B)$.

Recall from Section 2.4 (and [Lu3, L-HA]) that giving an E_n-algebra structure to $HH_{\mathcal{E}_n}(A, B) \cong RHom_A^{\mathcal{E}_n}\left(A, B \right)$ is equivalent to giving a structure of locally constant factorization algebra on D^n whose global section[22] are $RHom_A^{\mathcal{E}_n}\left(A, B \right)$. That is, we need to associate to any disk $U \subset D^n$ a chain complex $HH_{\mathcal{E}_n}\left(A, B \right)(U)$ naturally quasi-isomorphic to $RHom_A^{\mathcal{E}_n}\left(A, B \right)$ equipped with natural chain maps from

$$
(34) \quad \rho_{U_1,\dots,U_\ell,V} HH_{\mathcal{E}_n}\left(A, B \right)(U_1) \otimes \cdots \otimes HH_{\mathcal{E}_n}\left(A, B \right)(U_\ell) \to HH_{\mathcal{E}_n}\left(A, B \right)(V)
$$

for any pairwise disjoint (embedded) sub-disks U_i in a bigger disk V.

Let $\mathcal{A}$, $\mathcal{B}$ be the underlying locally constant factorization algebras on D^n associated to A and B given by Theorem 2.29 and still denote $f : \mathcal{A} \to \mathcal{B}$ the induced map of factorization algebras. In other words:

we assume from now on that A, B and f are given by locally constant factorization algebras as in Section 2.4.

---

[21] which depends on the map $f : A \to B$ even though it is not explicitly written in the notation

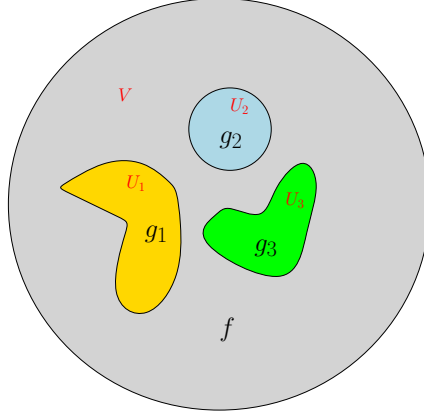[22] *i.e.*, its factorization homology over the whole disk D^n

FIGURE 1. The factorization algebra map $\mathcal{A}_{|V} \to \mathcal{B}_{|V}$ obtained by applying the relevant maps of modules g_1, g_2, g_3 (viewed as maps of factorizations algebras) and the E_n-algebra map $f : A \to B$ on the respective regions

Similarly, given any map of A-E_n-modules $g : A \to B$, by Proposition 5.6, we can *assume that g is given by a map* $g : \mathcal{A} \to \mathcal{B}$ *of (stratified) factorization algebras*, as well as, by Proposition 5.8 a map of (left) $\int_{S^{n-1}} A$-modules $g : \int_{D^n} A \to \int_{D^n} B$.

Remark 6.4 (*Sketch of the construction*). We first sketch the idea of the construction. For any sub-disk U_i, we can think of $HH_{\mathcal{E}_n}(A, B) \cong RHom_A^{\mathcal{E}_n}(A, B)$ as the space of stratified factorization algebras maps on the disk U_i (with a distinguished point $*_i$, see Proposition 5.6). Hence, given $g_1, \ldots, g_\ell \in HH_{\mathcal{E}_n}(A, B)$, we define the structure map (34) $\rho_{U_1, \ldots, U_\ell, V}(g_1, \ldots, g_\ell)$ to be the factorization algebra map which, to any sub-disk D inside a given U_i associates $g_i(D)$ and, to any disk D inside (a small neighborhood of) the complement of the U_i's associates $f(D)$. The family of those disks is a basis of all disks inside V, so that such a rule does define a factorization algebra map, which underlies a map of A-E-modules (see Remark 5.7). This is roughly described in Figure 1.

We now construct the locally constant factorization algebra on $\mathbb{R}^n$ we are looking for.

6.2.1. *Step 1: the underlying chain complexes.* For any open subset U, the restrictions $\mathcal{A}_{|U}$, $\mathcal{B}_{|U}$ are locally constant factorization algebras[23] on U, and $f_{|U} : \mathcal{A}_{|U} \to \mathcal{B}_{|U}$ a factorization algebra morphism. Thus, if U is a disk[24], $\mathcal{A}(U)$ $(\cong \int_U A)$ is an E_n-algebra and $f_{|U} = \int_U f$ makes $\mathcal{B}(U) \cong \int_U B$ an $\mathcal{A}(U)$-E_n-module[25]. In particular, given any point $*_U$ in U, the restrictions $\mathcal{A}_{|U}$, $\mathcal{B}_{|U}$ are canonically stratified factorization algebras on the pointed disk U and further define canonical objects in $\mathrm{Fac}_{U_{*_U}|\mathcal{A}_U}^{lc} \hookrightarrow \mathrm{Fac}_{U_{*_U}}^{lc,res}$, see Definition 5.3.

---

[23]quasi-isomorphic to A and B by definition

[24]that is an open set homeomorphic to a disk

[25]in this section we will write $\int_U A$ for $\mathcal{A}(U)$ viewed as an E_n-module over itself and reserve the notation $\mathcal{A}(U)$ when we think of it as an E_n-algebra

Thus to any open disk U, we can associate the following object of $k\text{-Mod}_\infty$:

$$(35) \qquad RHom_A^{\mathcal{E}_n}\Big(A, B\Big)(U) := RHom_{\mathcal{A}(U)}^{\mathcal{E}_n}\Big(\int_U A, \int_U B\Big)$$

Note that $RHom^{\mathcal{E}_n}\Big(A, B\Big)(U)$ is pointed since our starting map of E_n-algebras $f : A \to B$ induces a canonical element $\int_U f \in RHom_A^{\mathcal{E}_n}\Big(A, B\Big)(U)$.

6.2.2. *Step 2: the structure maps.* By Proposition 2.28, we only need to construct the factorization algebra $RHom_A^{\mathcal{E}_n}\Big(A, B\Big)(U)$ on the basis of opens subsets $\mathcal{CVX}$ consisting of all bounded open convex subsets of D^n. The basis $\mathcal{CVX}$ is stable by finite intersection and a factorizing cover. Note that if $U \in \mathcal{CVX}$ with center $*_U$, then (see Remark 6.3)

$$RHom_A^{\mathcal{E}_n}\Big(A, B\Big)(U) = RHom_{\int_{U \setminus \{*_U\}} A}^{left}\Big(\int_U A, \int_U B\Big)$$

which is *the mapping space between the associated stratified (in $*_U$) factorization algebras* $\mathcal{A}_{|U}$, $\mathcal{B}_{|U}$ corresponding to the module structures of $\int_U A$, $\int_U B$ as given by Proposition 5.6.

For pairwise disjoints disks $U_1, \ldots, U_r \in \mathcal{CVX}$ included in a larger disk $D \in \mathcal{CVX}$, we define the structure map

(36)

$$RHom_A^{\mathcal{E}_n}\Big(A, B\Big)(U_1) \otimes \cdots \otimes RHom_A^{\mathcal{E}_n}\Big(A, B\Big)(U_r) \overset{\rho_{U_1, \ldots, U_r, D}}{\longrightarrow} RHom_A^{\mathcal{E}_n}\Big(A, B\Big)(D)$$

as follows. Denote $*_1, \ldots, *_r$, the respective centers of the U_i's. First we use $U_1, \ldots, U_r$ to define the cover $\mathcal{U}_{U_1, \ldots, U_r, V}$ consisting of all opens V in D which

- either do not contain any $*_i$: $V \subset D \setminus \{*_1, \ldots, *_r\}$,
- or else is included in one of the U_i and is a neighborhood of $*_i$.

Maps of factorization algebras over D are uniquely determined by their value on $\mathcal{U}_{U_1, \ldots, U_r, V}$ since it is a factorizing cover of D. Let be given maps $g_i : \int_{|D_i} A \to \int_{|D_i} B$ of (left) $\mathcal{A}(U_i)$-modules ($i = 1 \ldots r$) and also denotes $g_i : \mathcal{A}_{|D_i} \to \mathcal{B}_{|D_i}$ the induced maps of stratified (at the point $*_i$) factorization algebras. We define $\rho_{U_1, \ldots, U_r, D}(g_1, \ldots, g_r)$ on an open $V \in \mathcal{U}_{U_1, \ldots, U_r, V}$ by:

$$(37) \qquad \rho_{U_1, \ldots, U_r, D}(g_1, \ldots, g_r)_{|V} = \begin{cases} f_{|V} & \text{if } V \subset D \setminus \{*_1, \ldots, *_r\}, \\ g_{i|V} & \text{if } *_i \in V \subset U_i. \end{cases}$$

Lemma 6.5. *The rule* $V \mapsto \rho_{U_1, \ldots, U_r, D}(g_1, \ldots, g_r)_{|V}$ *(given by Formula (37)), defines a map of factorization algebras*[26]:

$$\rho_{U_1, \ldots, U_r, D}(g_1, \ldots, g_r) \in Map_{Fac_{D_*}^{lc, res}}(\mathcal{A}_{|D}, \mathcal{B}_{|D}).$$

Proof. First we check that $\rho_{U_1, \ldots, U_r, D}(g_1, \ldots, g_r)$ defines a factorization algebra map. Since $\mathcal{U}_{U_1, \ldots, U_r, V}$ is a factorizing cover of D, we only need to check that it is compatible with the structure maps of $\mathcal{A}$ and $\mathcal{B}$. If all opens involved lies either in $D \setminus \{*_1, \ldots\}$ or contains a same point $*_i$, then the result is immediate since f and g_i are maps of factorization algebras. Now, assume $*_i \in V \subset U_i$ and that there are pairwise disjoint opens $V_1, \ldots, V_\ell \subset V$ (at most one of them can contain $*_i$).

---

[26]where $Fac_{D_*}^{lc, res}$ is given by Definition 5.3

Since g_i comes from a map of E_n-A-modules (with module structure induced by f), $g_{i|V_k} = f_{|V_k}$ whenever V_k does not contain $*_i$. It follows that the following diagram

$$
\begin{array}{ccc}
\bigotimes_{i=1\ldots\ell} \mathcal{A}(V_k) & \xrightarrow{\ \rho_{V_1,\ldots,V_\ell,V}\ } & \mathcal{A}(V) \\
{\scriptstyle \bigotimes_{i=1\ldots\ell} \rho_{U_1,\ldots,U_r,D}\,(g_1,\ldots,g_r)_{|V_k}} \downarrow & & \downarrow {\scriptstyle g_i} \\
\bigotimes_{i=1\ldots\ell} \mathcal{B}(V_k) & \xrightarrow{\ \rho_{V_1,\ldots,V_\ell,V}\ } & \mathcal{B}(V)
\end{array}
$$

is commutative, hence $\rho_{U_1,\ldots,U_r,D}\,(g_1,\ldots,g_r)$ is a map of (pre-)factorization algebras. $\qquad\square$

Lemma 6.6. *The induced (by Lemma 6.5) map*

$$
\int_D \rho_{U_1,\ldots,U_r,D}\,(g_1,\ldots,g_r) \in RHom_{\mathcal{A}(D)}^{\mathcal{E}_n}\Big(\int_D A, \int_D B\Big)
$$

is a map of $\mathcal{A}(D)$-E_n-modules.

In particular, we define the map (36) to be the global section $\int_D \rho_{U_1,\ldots,U_r,D}\,(g_1,\ldots,g_r)$ of the maps defined by formula (37).

Proof of Lemma 6.6. Passing to the global section in Lemma 6.5, we have the map $\rho_{U_1,\ldots,U_r,D}\,(g_1,\ldots,g_r)(D) : A \to B$ and we need to prove that it is a map of $\mathcal{A}(D)$-E_n-module. By Proposition 5.8, it is equivalent to prove that the induced map $\int_D \rho_{U_1,\ldots,U_r,D}\,(g_1,\ldots,g_r) : \int_D A \to \int_D B$ is a morphism of left $\int_{S^{n-1}} A$-modules (where the module structure is induced by f). Let $\tilde{D}$ be a closed sub-disk of D containing $U_1 \coprod \cdots \coprod U_r$. The open sub-set $D \setminus \tilde{D} \cong S^{n-1} \times (0,\epsilon')$ lies in the complement (in D) of the U_i's. Since $\int_{S^{n-1}} A$ is the section $\int_{S^{n-1}} A = \mathcal{A}\big(S^{n-1} \times (0,\epsilon')\big)$ (Theorem 2.20), we are left to prove that the map $\rho_{U_1,\ldots,U_r,D}\,(g_1,\ldots,g_r)$ restricted to $D \setminus \tilde{D} \cong S^{n-1} \times (0,\epsilon')$ is equivalent to f. This is an immediate consequence of the fact that $D \setminus \tilde{D} \subset D \setminus \{*_1,\ldots,*_i\}$ and $\big(\rho_{U_1,\ldots,U_r,D}\,(g_1,\ldots,g_r)\big)_{|\tilde{D}\subset D\setminus\{*_1,\ldots,*_i\}} = f$ as given by construction (37). $\qquad\square$

Remark 6.7. Let us consider the case of the inclusion of the empty set $\emptyset$ inside a disk D. Unwinding the definition of the structure map

$$
\rho_{\emptyset,D} : k \cong RHom_A^{\mathcal{E}_n}\big(A,B\big)(\emptyset) \longrightarrow RHom_A^{\mathcal{E}_n}\big(A,B\big)(D)
$$

we see immediately that $\rho_{\emptyset,D}(1) = \int_D f$, in other words 1 is mapped to the base point of $RHom^{\mathcal{E}_n}\big(A,B\big)(D)$.

A straightforward computation also shows that

$$
\rho_{U_1,\ldots,U_r,D}\Big(\int_{U_1} f,\ldots,\int_{U_r} f\Big) = \int_D f.
$$

6.2.3. *Step 3: the global structure.* The cochain complexes $U \mapsto RHom_A^{\mathcal{E}_n}\big(A,B\big)(U) \cong Hom_{\mathcal{A}(U)}^{\mathcal{E}_n}\big(\int_U A, \int_U B\big)$ are equipped with the structure maps (36) (given by formula (37) and Lemma 6.6). These maps assemble to form a locally constant factorization algebra over $\mathbb{R}^n$, yielding an E_n-algebra structure to $RHom_A^{\mathcal{E}_n}\big(A,B\big)$. This is the content of the following result:

Theorem 6.8. *Let $f : A \to B$ be a map of E_n-algebras[27].*

(1) *The structure maps (41) $\rho_{U_1,\dots,U_r,V}$ (given by step 2 above) make $U \mapsto RHom_A^{\mathcal{E}_n}(A, B)(U)$ a locally constant factorization algebra on $\mathbb{R}^n$ whose global section are naturally equivalent to $RHom_A^{\mathcal{E}_n}(A, B)$.*

(2) *In particular $HH_{\mathcal{E}_n}(A, B) \cong RHom_A^{\mathcal{E}_n}(A, B)$ inherits a natural E_n-algebra structure (with unit given by f).*

(3) *Let $g : B \to C$ be another map of E_n-algebrass[28]. The (derived) functor of composition of E_n-modules homomorphisms*

$$RHom_A^{\mathcal{E}_n}(A, B) \otimes RHom_B^{\mathcal{E}_n}(B, C) \overset{\circ}{\longrightarrow} RHom_A^{\mathcal{E}_n}(A, C)$$

is a homomorphism of E_n-algebras[29].

(4) *Let $h : C \to D$ be an E_n-algebra map. The canonical map*

$$RHom_A^{\mathcal{E}_n}(A, B) \otimes RHom_C^{\mathcal{E}_n}(C, D) \longrightarrow RHom_{A \otimes C}^{\mathcal{E}_n}(A \otimes C, B \otimes D)$$

is a homomorphism of E_n-algebras.

The naturality (in B) of the E_n-algebra structure of $RHom_A^{\mathcal{E}_n}(A, B)$ means that, given a morphism $\phi : B \to B'$ of E_n-algebras, the induced map

$$\phi_* : RHom_A^{\mathcal{E}_n}(A, B) \to RHom_A^{\mathcal{E}_n}(A, B') \quad \text{(given by } g \mapsto \phi \circ g\text{)}$$

is an E_n-algebra morphism. Here, the A-module structure of B is of course given by the E_n-algebra morphism $\phi \circ f : A \to B'$. Similarly, the naturality in A means that, given a morphism $\psi : A' \to A$ of E_n-algebras, the induced map

$$\psi_* : RHom_A^{\mathcal{E}_n}(A, B) \to RHom_{A'}^{\mathcal{E}_n}(A', B) \quad \text{(given by } g \mapsto g \circ \psi\text{)}$$

is an E_n-algebra morphism.

Proof of Theorem 6.8. Since the global section $\mathcal{F}(\mathbb{R}^n)$ of a locally constant factorization algebra $\mathcal{F}$ on $\mathbb{R}^n$ is an E_n-algebra (Theorem 2.29), the second statement is an immediate consequence of the first one.

We now prove the first one. Proposition 2.28 implies that we need only to check the axioms of a locally constant factorization algebra on the basis of opens $\mathcal{CVX}$.

First we prove the naturality of the structure maps (36) with respect to the inclusion of open convex disks (in other words we check the prefactorization algebra axiom). That is we need to check that for a family of pairwise disjoints disks $U_1, \dots U_r \in \mathcal{CVX}$ inside a disk $V \in \mathcal{CVX}$ and families $W_1^j \dots W_{i_j}^j$ of pairwise disjoints convex disks inside U_j (for $j = 1 \dots r$) we have

$$(38) \qquad \rho_{U_1,\dots,U_r,V}\left(\rho_{W_1^1,\dots,W_{i_1}^1,U_1}, \dots, \rho_{W_1^r,\dots,W_{i_r}^r,U_r}\right) = \rho_{W_1^1,\dots,W_{i_1}^1,\dots,W_1^r,\dots,W_{i_r}^r,V}.$$

We write respectively $*_i$ and $*_{i_j}^i$ for the centers of the U_i's and $W_{i_j}^i$'s. Recall that the structure maps in the above identity (38) are obtained by applying the construction (37) on the relevant opens subsets. Thus the right hand side of (38) is the global section of the map of factorization algebras which is equal to $g_{i_k}^j$ on a open $W_{i_k}^j$ which contains $*_{i_k}^j$ and f on opens lying in the complement of $*_{i_\ell}^s$.

To evaluate the left hand side, we first define a specific cover of V as follows. For each U_i $(i = 1 \ldots r)$, we choose a convex closed sub-disk of U_i which contains each W_j^i and $*_i$. We write col_i for its complement in U_i. Then, we have a cover of V given by the U_i's and $U_\partial = V \setminus \left(\coprod_{i=1}^r (U_i - col_i) \right)$. The left hand side of identity (38) is determined by its restriction on the cover (see § 2.4 and [CG, G2]). By construction (37), the map $\rho_{U_1,\ldots,U_r,V\,|U_\partial}$ is equal to $f_{|U_\partial}$. Let $i \in \{1,\ldots,r\}$ and D be an opens in U_i. If $*_j^i \in D \subset W_j^i$, by construction (37), the composition

$$\rho_{U_1,\ldots,U_r,V}\left(\rho_{W_1^1,\ldots,W_{i_1}^1,U_1}(g_1^1,\ldots,g_{i_1}^1),\ldots,\rho_{W_1^r,\ldots,W_{i_r}^r,U_r}(g_1^r,\ldots,g_{i_r}^r)\right)_{|D}(V)$$

on the open V is given by $g_j^i : \mathcal{A}_{|W_j^k} \to \mathcal{B}_{|W_j^k}$, that is, is equal to $\int_V g_j^i$. While if $D \subset U_i \setminus \{*_1^i,\ldots,*_{j_i}^i\}$, then this composition is equal to $\int_V f$. Note that the composition agrees with the map induced by f on the intersection of the U_i's with U_∂. It follows that the left hand side of identity (38) is the unique factorization algebra map which coincides with g_j^i on each open subset of W_j^i containing $*_j^i$ and coincides with f on opens which do not contains any $*_j^i$. It is thus equal to the right hand side of (38). We have proved that the structure maps $\rho_{U_1,\ldots,U_r,V}$ satisfies the associativity condition of a prefactorization algebra. They also satisfy the symmetry condition since they are independent of any ordering of the opens $U_1,\ldots,U_r$.

It remains to check that $U \mapsto RHom_A^{\mathcal{E}_n}\left(A,B\right)(U)$ is locally constant. Since the factorization algebras $\mathcal{A}$ and $\mathcal{B}$ are locally constant, the natural maps $\int_U A \to \int_V A$ and $\int_U B \to \int_V B$ are equivalences for any embedding $U \hookrightarrow V$ of a disk U inside a bigger disk V. By definition we have

$$RHom_A^{\mathcal{E}_n}\left(A,B\right)(U) \;\cong\; Hom_{\mathcal{A}(U)}^{\mathcal{E}_n}\left(\int_U A, \int_U B\right),$$
$$RHom_A^{\mathcal{E}_n}\left(A,B\right)(V) \;\cong\; Hom_{\mathcal{A}(V)}^{\mathcal{E}_n}\left(\int_V A, \int_V B\right).$$

By definition, for any $g \in RHom_A^{\mathcal{E}_n}(A,B)(U)$, the map

$$\rho_{U,V} : Hom_{\mathcal{A}(U)}^{\mathcal{E}_n}\left(\int_U A, \int_U B\right) \longrightarrow Hom_{\mathcal{A}(V)}^{\mathcal{E}_n}\left(\int_V A, \int_V B\right)$$

applied to g is induced by a map of factorization algebras $\rho_{U,V}(g) : \mathcal{A}_{|V} \to \mathcal{B}_{|V}$ whose restriction to U is just g. It follows that the following diagram is commutative

$$
\begin{array}{ccc}
\int_V A & \xrightarrow{\;\rho_{U,V}(g)\;} & \int_V B \\
\simeq \big\uparrow & & \big\uparrow \simeq \\
\int_U A & \xrightarrow{\quad g \quad} & \int_U B
\end{array}
$$

for all $g \in RHom_A^{\mathcal{E}_n}(A,B)(U)$. Since the vertical maps are equivalences and independent of g, it follows that $\rho_{U,V} : Hom_{\mathcal{A}(U)}^{\mathcal{E}_n}\left(\int_U A, \int_U B\right) \to Hom_{\mathcal{A}(V)}^{\mathcal{E}_n}\left(\int_V A, \int_V B\right)$

is an equivalence. Note in particular that, taking $V = \mathbb{R}^n$, we have canonical equiv-
alences

$$RHom_A^{\mathcal{E}_n}\big(A, B\big)(U) \cong Hom_{\mathcal{A}(U)}^{\mathcal{E}_n}\Big(\int_U A, \int_U B\Big)$$

$$\cong Hom_{\mathcal{A}(\mathbb{R}^n)}^{\mathcal{E}_n}\Big(\int_{\mathbb{R}^n} A, \int_{\mathbb{R}^n} B\Big) \cong RHom_A^{\mathcal{E}_n}\big(A, B\big)$$

for any disk U in $\mathbb{R}^n$.

A map of E_n-algebras $g : B \to C$ induces a canonical object in $RHom_B^{\mathcal{E}_n}(B, C)$
given by g itself. Thus the naturality of the E_n-algebra structure (claimed in as-
sertion **(2)**) is in fact a consequence of the assertion **(3)** in the Theorem (that
we will prove below). To finish the proof of claims **(1)**, **(2)** in the Theorem we
need to see that the canonical element $f \in RHom_A^{\mathcal{E}_n}(A, B)$ is a unit. Indeed, let
$U_1, \ldots, U_r, V$ be a finite family of pairwise disjoints convex disks inside a bigger
bounded convex open set D, and $g_i \in RHom_A^{\mathcal{E}_n}(A, B)(U_i)$ ($i = 1 \ldots r$). Denote
$*_1, \ldots, *_r, *_V$ the respective centers of U_i's and V. Let also $f \in RHom_A^{\mathcal{E}_n}(A, B)(V)$
be the canonical element induced by f. By definition (see construction (37))
$\rho_{U_1,\ldots,U_r,V,D}(g_1, \ldots, g_r, f)$ is the factorization algebra map whose values on any
open subset $W \subset V \setminus \{*_1, \ldots, *_r, *_V\}$ is given by (the restriction to V of) f, whose
value on any open subset $*_i \in W \subset U_i$ is given by g_i and its value on $*_V \in W \subset V$
is again given by f. It follows that this map is equal to f on all V and thus we get

$$\rho_{U_1,\ldots,U_r,V,D}(g_1, \ldots, g_r, f) = \rho_{U_1,\ldots,U_r,D}(g_1, \ldots, g_r).$$

This proves that f is a unit for the E_n-algebra structure of $RHom_A^{\mathcal{E}_n}(A, B)$.

We now prove statement **(3)**. Since the B-module structure of C is given by the
E_n-algebra map $g : B \to C$, the (derived) composition of maps $RHom_A^{\mathcal{E}_n}\big(A, B\big) \otimes$
$RHom_B^{\mathcal{E}_n}\big(B, C\big) \xrightarrow{\circ} Hom_{k\text{-}\mathrm{Mod}_\infty}\big(A, C\big)$ naturally lands in $RHom_A^{\mathcal{E}_n}\big(A, C\big)$, where
C is endowed with the A-module structure induced by the E_n-algebra morphism
$g \circ f : A \to C$. Since the tensor product of E_n-algebras is induced by the tensor
products of (locally constant) factorization algebras, it remains to prove that, for
any family $U_1, \ldots, U_r$ of pairwise disjoint open disks included inside a bigger disk
D, the following diagram
(39)

$$\bigotimes_{i=1}^r \Big(RHom_A^{\mathcal{E}_n}\big(A, B\big)(U_i) \otimes RHom_B^{\mathcal{E}_n}\big(B, C\big)(U_i)\Big) \xrightarrow{\overset{r}{\underset{i=1}{\bigotimes}} \circ} \bigotimes_{i=1}^r RHom_A^{\mathcal{E}_n}\big(A, C\big)(U_i)$$

$$\left\downarrow{\scriptstyle \rho_{U_1,\ldots,U_r,D}^{\otimes 2}} \qquad\qquad\qquad\qquad\qquad\qquad \left\downarrow{\scriptstyle \rho_{U_1,\ldots,U_r,D}}\right.$$

$$RHom_A^{\mathcal{E}_n}\big(A, B\big)(D) \otimes RHom_B^{\mathcal{E}_n}\big(B, C\big)(D) \xrightarrow{\quad\circ\quad} RHom_A^{\mathcal{E}_n}\big(A, C\big)(D)$$

is commutative in $k\text{-}\mathrm{Mod}_\infty$. Let be given $\phi_i \in RHom_A^{\mathcal{E}_n}\big(A, B\big)(U_i)$ and $\psi_i \in$
$RHom_B^{\mathcal{E}_n}\big(B, C\big)(U_i)$. We keep denoting $\phi_i : \mathcal{A}_{|U_i} \to \mathcal{B}_{|U_i}$ and $\psi_i : \mathcal{B}_{|U_i} \to \mathcal{C}_{|U_i}$ the
induced maps of factorization algebras. The result of the two compositions in dia-
gram (39), namely $\rho_{U_1,\ldots,U_r,D}(\psi_1, \ldots, \psi_r) \circ \rho_{U_1,\ldots,U_r,D}(\phi_1, \ldots, \phi_r)$ and $\rho_{U_1,\ldots,U_r,D} \circ$
$\big(\otimes_{i=1}^r \psi_i \circ \phi_i\big)$ are both global sections over D of factorization algebras morphisms.

It is thus enough to prove that the underlying diagram of factorizations algebras

(40)
$$\mathcal{A}_{|D} \underset{\rho_{U_1,\ldots,U_r,D}(\psi_1,\ldots,\psi_r)_{|D} \circ \rho_{U_1,\ldots,U_r,D}(\phi_1,\ldots,\phi_r)_{|D}}{\overset{\rho_{U_1,\ldots,U_r,D}\left(\otimes_{i=1}^r \psi_i \circ \phi_i\right)_{|D}}{\rightleftarrows}} \mathcal{C}_{|D}$$

is commutative (here the structure maps are given by construction (37)).

It is sufficient to prove the result on the stable under finite intersection factorizing basis $\mathcal{U}_{U_1,\ldots,U_r,D}$ given in step **2** (§ 6.2.2). The upper arrow in diagram (40) is simply the factorization algebra map which is equal to $\psi_i \circ phi_i$ on any open subset $*_i \subset W \subset U_i$ and is equal to $g \circ f$ for any other $W \in \mathcal{U}_{U_1,\ldots,U_r,D}$. On the other hand the lower map in diagram (40) is the composition of two factorizations algebras maps: one of which being given by ϕ_i on any open subset $*_i \subset W \subset U_i$ and f on any other $W \in \mathcal{U}_{U_1,\ldots,U_r,D}$; while the other one is given by ψ_i on any open subset $*_i \subset W \subset U_i$ and g on any other $W \in \mathcal{U}_{U_1,\ldots,U_r,D}$. The commutativity of diagram (40) follows on $\mathcal{U}_{U_1,\ldots,U_r,D}$ and thus diagram (39) also commutes.

It remains to prove statement (4) in Theorem 6.8 which is almost trivial: let $h : C \to D$ be an E_n-algebra map and denote $\mathcal{C}, \mathcal{D}$ the associated factorization algebras on $\mathbb{R}^n$. By [L-HA, Theorem 5.3.3.1],

$$\int_{U \cup V} A \cong \int_U A \otimes \int_V A$$

for any E_n-algebra A and disjoint open sets U, V. Thus, the factorization algebra associated (in Proposition 5.8) to $A \otimes C$ is given by $U \mapsto \mathcal{A}(U) \otimes \mathcal{C}(U)$. For any pairwise disjoints open convex disks $U_1, \ldots, U_r$ included in a bigger convex disk $V \subset D^n$, and maps $g_i \in RHom_A^{\mathcal{E}_n}(A, B)(U_i)$, $g_i' \in RHom_C^{\mathcal{E}_n}(C, D)(U_i)$ $(i = 1 \ldots r)$, the map

$$\rho_{U_1,\ldots,U_r,V}\left(\left(g_1 \otimes \cdots \otimes g_r\right) \otimes \left(g_1' \otimes \cdots \otimes g_r'\right)\right) \in \left(RHom_A^{\mathcal{E}_n}(A, B) \otimes RHom_C^{\mathcal{E}_n}(C, D)\right)(V)$$
$$\cong RHom_A^{\mathcal{E}_n}(A, B)(V) \otimes RHom_C^{\mathcal{E}_n}(C, D)(V)$$

is the map obtained as the global section of a map of factorization algebras

$$\rho_{U_1,\ldots,U_r,V}\left(\left(g_1 \otimes \cdots \otimes g_r\right) \otimes \left(g_1' \otimes \cdots \otimes g_r'\right)\right)_{|V} : \mathcal{A}_{|V} \otimes \mathcal{C}_{|V} \to \mathcal{B}_{|V} \otimes \mathcal{D}_{|V}$$

as constructed in § 6.2.2. In particular, for any $i \in \{1, \ldots, r\}$, its value in any open subset $*_i \in W \subset U_i$ is given by the map $g_i \otimes g_i' : \mathcal{A}_{|U_i} \otimes \mathcal{C}_{|U_i} \to \mathcal{B}_{|U_i} \otimes \mathcal{D}_{|U_i}$. Further, its value on any other open subset $W \in \mathcal{U}_{U_1,\ldots,U_r,V}$ is given by $f_{|V} \otimes g_{|V}$ (here we use freely the notations introduced in § 6.2.2 to define the structure maps $\rho_{U_1,\ldots,U_r,V}$).

Hence, the map $\rho_{U_1,\ldots,U_r,V}\left(\left(g_1 \otimes \cdots \otimes g_r\right) \otimes \left(g_1' \otimes \cdots \otimes g_r'\right)\right)_{|V}$ identifies, on (the factorizing and stable by finite intersection) cover $\mathcal{U}_{U_1,\ldots,U_r,V}$, with the map

obtained by evaluating the composition

$$\Big(\bigotimes_{i=1}^{r} RHom_A^{\mathcal{E}_n}(A,B)(U_i)\Big) \otimes \Big(\bigotimes_{i=1}^{r} RHom_C^{\mathcal{E}_n}(C,D)(U_i)\Big)$$

$$\longrightarrow \bigotimes_{i=1}^{r} RHom_{A\otimes C}^{\mathcal{E}_n}(A\otimes C, B\otimes D)(U_i)$$

$$\overset{\rho_{U_1,\dots,U_r,V}}{\longrightarrow} RHom_{A\otimes C}^{\mathcal{E}_n}(A\otimes C, B\otimes D)(V)$$

at the tensor product $(g_1\otimes\cdots\otimes g_r)\otimes(g_1'\otimes\cdots\otimes g_r')$. This proves that the canonical maps

$$RHom_A^{\mathcal{E}_n}(A,B)(V) \otimes RHom_C^{\mathcal{E}_n}(C,D)(V) \longrightarrow RHom_{A\otimes C}^{\mathcal{E}_n}(A\otimes C, B\otimes D)(V)$$

assembles into a map of factorization algebras and, consequently,

$$RHom_A^{\mathcal{E}_n}(A,B) \otimes RHom_C^{\mathcal{E}_n}(C,D) \longrightarrow RHom_{A\otimes C}^{\mathcal{E}_n}(A\otimes C, B\otimes D)$$

is a homomorphism of E_n-algebras. $\square$

Remark 6.9. The E_n-algebra structure given by Theorem 6.8 is in fact the solution of a universal property as will be given by Proposition 6.22 below which identifies $HH_{\mathcal{E}_n}(A,B)$ with the centralizer of the map $f : A \to B$.

Example 6.10. Assume $n = 1$, then

$$HH_{\mathcal{E}_1}(A,B) \cong RHom_A^{\mathcal{E}_1}(A,B) \cong RHom_{\int_{S^0} A}(A,B) \cong RHom_{A\otimes A^{op}}(A,B)$$

is the standard Hochschild cohomology of the algebra A with value in the algebra B. It is straightforward that the E_1-structure given by Theorem 6.8 is induced on the standard Hochschild complex by the usual cup-product [Ge].

Example 6.11. Assume $A = k$ the ground ring and let $f : k \to B$ be the unit map. We have a canonical equivalence $RHom_k^{\mathcal{E}_n}(k,B) \cong B$ in k-Mod$_\infty$. This equivalence is in fact an *equivalence of E_n-algebras*[30]. Since $f : k \to B$ is the unit map, it is immediate from the definition of the structure maps (41) to check that the locally constant factorization algebra structure of $RHom_k^{\mathcal{E}_n}(k,B)$ is the one of $\mathcal{B}$, the locally constant factorization algebra on $\mathbb{R}^n$ associated to B (in § 2.4).

Remark 6.12. Since the factorization algebra constructed by Theorem 6.8 is locally constant, in particular, *its value $RHom_A^{\mathcal{E}_n}(A,B)(U)$ on any (non-necessarily convex) disk U is $RHom_{\mathcal{A}(U)}^{\mathcal{E}_n}\left(\int_U A, \int_U B\right)$* as asserted in step 1 (§ 6.2.1).

One can describe directly the structure maps $\rho_{U_1,\dots,U_r,D}$ associated to pairwise disjoint subdisks $U_1,\dots,U_r$ of a disk D (that is without further covering them by convex subsets). This can be done as follows. First, we use $U_1,\dots,U_n$ to see the factorization algebra $\mathcal{A}$ restricted to D, denoted $\mathcal{A}_{|D}$, as obtained by gluing together $r+1$-factorization algebras on D (see § 2.3 and [CG]). Note that the $\mathcal{A}(U)$-E_n-module structure on $\int_U A$ allows us to see $\mathcal{A}_{|U}$ as a stratified factorization algebra on U with a closed strata given by a point $*_i$ (or a sub-disk); different choices of points leads to canonically equivalent $\mathcal{A}_{|U}$-modules structures. We can choose a collar c_i in the neighborhood of the boundary (in D) of each U_i such that $c_i \cong S^{n-1} \times (0,\epsilon)$ for a homeomorphism induced by a homeomorphism $U_i \cong D^n$; for instance we just

---

[30]where the left hand side is endowed with the E_n-algebra structure given by Theorem 6.8

choose c_i to be the complement of the point $*_i \in U_i$ (or a suitable sub-disk). This way we get a connected open set

$$U_\partial := D \setminus \Big(\coprod_{i=1}^{r} (U_i - col_i) \Big)$$

(the notation ∂ is meant to suggest that U_∂ is the boundary of $\coprod U_i$ in D; below we will sometimes refer to it using this terminology). By definition, the U_i's and U_∂ cover D, hence the factorization algebra $\mathcal{A}_{|D}$ is obtained as the gluing of the restricted factorizations algebras $\mathcal{A}_{|U_1}, \ldots, \mathcal{A}_{|U_r}$ and $\mathcal{A}_{|U_\partial}$.

Let be given maps $g_i : \int_{|D_i} A \to \int_{|D_i} B$ of (left) $\mathcal{A}(U_i)$-modules ($i = 1 \ldots r$) and also denotes $g_i : \mathcal{A}_{|D_i} \to \mathcal{B}_{|D_i}$ the induced maps of (stratified) factorization algebras. We also denote $f_\partial : \mathcal{A}_{|U_\partial} \to \mathcal{B}_{|U_\partial}$ the restriction of $f : \mathcal{A} \to \mathcal{B}$ to U_∂.

Lemma 6.13. *The family $(g_1, \ldots, g_r, f_\partial)$ of maps of factorization algebras glues together to define a map*

$$\rho_{U_1,\ldots,U_r,D}(g_1, \ldots, g_r) \in RHom(\mathcal{A}_{|D}, \mathcal{B}_{|D})$$

which is independent in k-Mod_∞ of the choices (of collars) involved. Further, on global sections, the induced map $\int_D \rho_{U_1,\ldots,U_r,D}(g_1, \ldots, g_r) : \int_D A \to \int_D B$ is a map of $\mathcal{A}(D)$-E_n-modules. The induced map

$$(41) \quad \rho_{U_1,\ldots,U_r,D} : RHom_A^{\mathcal{E}_n}\Big(A, B\Big)(U_1) \otimes \cdots \otimes RHom_A^{\mathcal{E}_n}\Big(A, B\Big)(U_r)$$
$$\longrightarrow RHom_A^{\mathcal{E}_n}\Big(A, B\Big)(D)$$

is the map given by the factorization algebra structure of Theorem 6.8.

Proof of Lemma 6.13. Note that all triples intersections in the family $(U_1, \ldots, U_r, U_\partial)$ are empty and that the only non-empty intersections are those of the form $U_i \cap U_\partial = col_i \cong S^{n-1} \times (0, \epsilon)$. Hence, by definition of the gluing of factorization algebras, we only have to check that the maps g_i and f_∂ are equivalent on $\mathcal{A}_{|(U_i \cap U_\partial)}$. By assumption, the map $g_i : \int_{U_i} A \to \int_{U_i} B$ is a map of $\mathcal{A}(U_i)$-modules. Then Proposition 5.8 (and Theorem 2.20) implies that the map of factorization algebras $g_i : \mathcal{A}_{|col_i} \to \mathcal{B}_{|col_i}$ is equivalent to the map induced by the $\mathcal{A}(U_i)$-module structure of $\int_{U_i} B \cong B$. Since this module structure is given by $f : \mathcal{A} \to \mathcal{B}$, it follows that $\big(g_i\big)_{|col_i}$ is equivalent to $\big(f_\partial\big)_{|col_i}$. Hence the collection $(g_1, \ldots, g_r, f_\partial)$ assembles to give an object in $RHom(\mathcal{A}_{|D}, \mathcal{B}_{|D})$. Further, we also just proved, that for any choice of collar col_i' in the disk U_i, the value of g_i on $\mathcal{A}_{col'_i}$ is given by f. It is thus independent of the choice of the collar. In order to check it induces the same map as Theorem 6.8, it is sufficient to check that the underlying maps of factorizations algebras agrees. For this, it is further sufficient to do it on the cover of D obtained by taking only convex open subsets which are required to belong to either one of the U_i or to U_∂, for which the result follows by definition. $\square$

6.2.4. *The parametrized factorization algebra structure on Hochschild cohomology of E_n-algebras.* The E_n-algebra structure given in Theorem 6.8 is given by a factorization algebra structure (by Theorem 2.29). In view of Proposition 2.25, it can also be obtained as a *parametrized* locally constant factorization algebra (see Definition 2.24), that is a locally constant algebra over the operad $N(\mathrm{Disk}(M)')$. This structure is rather easy to describe as we now explain.

Let $f : A \to B$ be an E_n-algebra map. As before we may assume that the map is induced by a map $f : \mathcal{A} \to \mathcal{B}$ of locally constant factorization algebras over $\mathbb{R}^n$. From Definition 2.24, we see that we need to associate to any open embedding $\phi : \mathbb{R}^n \to \mathbb{R}^n$ a chain complex $HH_{\mathcal{E}_n}(A, B)(\phi)$. We set

$$(42) \quad HH_{\mathcal{E}_n}(A, B)(\phi) := RHom_{\mathcal{A}(\phi(\mathbb{R}^n))}^{\mathcal{E}_n}(A, B)(U) \cong RHom_{\int_{\partial U} A}^{left}\left(\int_U A, \int_U B\right)$$

where $\partial U := \phi\big(\mathbb{R}^n \setminus D(0, 1)\big)$ is the image by ϕ of the complement of a closed (bounded) Euclidean disk centered at 0. Note that any different choice of radius yields canonically equivalent chain complexes since $\mathcal{A}$, $\mathcal{B}$ are locally constant.

Now, we need to define structure maps associated to any open embedding $h : \coprod_{i=1}^r \mathbb{R}^n \to \mathbb{R}^n$ such that $\psi \circ h = \coprod_{i=1}^r \phi_i : \coprod_{i=1}^r \mathbb{R}^n \to M$. The structure map

$$\rho_{\phi_1,\dots,\phi_r,\psi}^h : HH_{\mathcal{E}_n}(A, B)(\phi_1) \otimes \cdots \otimes HH_{\mathcal{E}_n}(A, B)(\phi_r) \longrightarrow HH_{\mathcal{E}_n}(A, B)(\psi)$$

is defined exactly in the same way as in Remark 6.12 and in particular Lemma 6.13. Here, we can use the canonical collars given by the complement of a disk centered at 0 inside each disk $\mathbb{R}^n$. Then, using this slight alternative definition of factorization algebras, one proves (in a similar way) the obvious analogue of Theorem 6.8. A proof similar to the one of Theorem 6.8 yields

Proposition 6.14. *Let $f : A \to B$ be a map of E_n-algebras*[31].

(1) *The structure maps (41) $\rho_{\phi_1,\dots,\phi_r,\psi}^h$ make $U\phi \mapsto RHom_A^{\mathcal{E}_n}(A, B)(\phi)$ a locally constant parametrized factorization algebra on $\mathbb{R}^n$ whose global section are naturally equivalent to $RHom_A^{\mathcal{E}_n}(A, B)$.*

(2) *This parametrized factorisation algebra is equivalent to the one of Theorem 6.8 under the equivalence of Proposition 2.25.*

(3) *the composition and tensor product of endomorphisms are maps of (locally constant) parametrized factorization algebras.*

6.2.5. *The case of E_∞-algebras again.* If $f : A \to B$ is a map of E_∞-algebras, then Theorem 6.8 and Proposition 6.2 give an E_n-algebra structure to $CH^{S^n}(A, B)$. The latter has also an E_n-algebra structure given by Theorem 4.12. The following result shows that these two structures are the same.

Proposition 6.15. *Let $f : A \to B$ be a map of E_∞-algebra and let B be endowed with the induced A-E_∞-module structure. Then the natural equivalence $HH_{\mathcal{E}_n}(A, B) \cong CH^{S^n}(A, B)$ given by Proposition 6.2 is an equivalence of E_n-algebras*[32].

Proof. The proof of Proposition 6.2 shows that we have equivalences

(43)

$$HH_{\mathcal{E}_n}(A, B) \cong RHom_{\int_{S^{n-1}} A}^{left}\left(\int_{D^n} A, B\right) \cong RHom_{CH_{S^{n-1}}(A)}^{left}\big(CH_{D^n}(A), B\big)$$

$$\cong Hom_{CH_{D^n}(A)}^{\mathcal{E}_n}\big(CH_{D^n}(A), B\big).$$

---

[31] which we may assume to be given by a map $f : \mathcal{A} \to \mathcal{B}$ of factorization algebras, see § 2.4

[32] where the left hand-side is the E_n-algebra given by Theorem 6.8 and the right hand side is the E_n-algebra given by Theorem 4.12

By Theorem 3.13, we have natural (in spaces and E_∞-algebras) equivalences $\int_U A \cong CH_U(A)$ and, by the value on a point axiom in Definition 3.25, a canonical equivalence $CH_U(B) \cong B$. We can thus define a rule $U \mapsto Hom_{CH_U(A)}^{\mathcal{E}_n}(CH_U(A), B)$ and structure maps (for $U_1, \ldots, U_r$ pairwise disjoints convex opens included in a larger bounded convex open D)

$$(44) \quad \rho_{U_1,\ldots,U_r,D} : Hom_{CH_{U_1}(A)}^{\mathcal{E}_n}\big(CH_{U_1}(A), B\big) \otimes \cdots \otimes Hom_{CH_{U_r}(A)}^{\mathcal{E}_n}\big(CH_{U_r}(A), B\big)$$
$$\longrightarrow Hom_{CH_D(A)}^{\mathcal{E}_n}\big(CH_D(A), B\big)$$

defined exactly as the structure maps (36) (in step 2, § 6.2.2) for $RHom_A^{\mathcal{E}_n}(A, B)(U)$. Then the proof of Theorem 6.8 applies *mutatis mutandis* to prove that $U \mapsto Hom_{CH_U(A)}^{\mathcal{E}_n}(CH_U(A), B)$ is a *locally constant factorization algebra on D^n* and further, that the equivalences

$$Hom_{\mathcal{A}(U)}^{\mathcal{E}_n}\Big(\int_U A, \int_U B\Big) \cong Hom_{CH_U(A)}^{\mathcal{E}_n}\big(CH_U(A), B\big)$$

(induced by Theorem 3.13) are equivalences of factorization algebras.

Now, for any collar $\partial U \cong S^{n-1} \times (0, \epsilon)$ inside a disk U, we have natural equivalences

$$
\begin{aligned}
Hom_{CH_U(A)}^{\mathcal{E}_n}\big(CH_U(A), B\big) &\cong RHom_{CH_{\partial U}(A)}^{left}\big(CH_U(A), B\big) \\
&\cong RHom_A\Big(A, RHom_{CH_{\partial U}(A)}^{left}\big(CH_U(A), B\big)\Big) \\
&\cong RHom_A\Big(CH_U(A) \underset{CH_{\partial U}(A)}{\overset{\mathbb{L}}{\otimes}} A, B\Big) \\
&\cong RHom_A\Big(CH_{U/\partial U}(A), B\Big) \cong CH^{U/\partial U}(A, B)
\end{aligned}
$$

where the last equivalences are by the excision axiom (Definition 3.25) and definition of Hochschild cochains. Recall from the proof of Proposition 6.2 that, for $U = D^n$, the above equivalences and the equivalences (43) are precisely the natural equivalence $HH_{\mathcal{E}_n}(A, B) \cong CH^{S^n}(A, B)$ of Proposition 6.2. Hence, we are left to prove Proposition 6.15 replacing $HH_{\mathcal{E}_n}(A, B)$ with $CH^{S^n}(A, B)$ endowed with the E_n-algebra structure given by the locally constant factorization algebra structure.

By functoriality of Hochschild chains, we have natural maps of E_∞-algebras

$$(45) \quad \bigotimes_{i=1}^{r} B \cong \bigotimes_{i=1}^{r} CH_{U_i}(B) \cong CH_{\bigcup_{i=1}^{r} U_i}(B) \overset{(\bigcup_{i=1}^{r} U_i \hookrightarrow V)_*}{\longrightarrow} CH_V(B) \cong B$$

Further, we have pinching maps $D \overset{p_{U_1,\ldots,U_r,D}}{\longrightarrow} \bigvee_{i=1}^{r} \big(U_i/\partial U_i\big)$ obtained by collapsing $D \setminus \big(\bigcup_{i=1}^{r} U_i \setminus \partial U_i\big)$ to a point. By functoriality of Hochschild cochains, the pinching maps yield a map

$$(46) \quad \bigotimes_{i=1}^{r} CH_{U_i}(A) \cong CH_{\bigcup_{i=1}^{r} U_i}(A) \overset{(p_{U_1,\ldots,U_r,D})_*}{\longrightarrow} CH_{\bigvee_{i=1}^{r} \big(U_i/\partial U_i\big)}(A).$$

Using the last two maps, we get the composition

$$(47) \quad \bigotimes_{i=1}^{r} RHom_{CH_{\partial U_i}(A)}^{left}\big(CH_{U_i}(A), B\big) \longrightarrow RHom_{\otimes_{i=1}^{r} A}\big(\bigotimes_{i=1}^{r} CH_{U_i}(A), \bigotimes_{i=1}^{r} B\big)$$

$$\longrightarrow RHom_{\otimes_{i=1}^{r} A}\big(\bigotimes_{i=1}^{r} CH_{U_i}(A), B\big) \cong RHom_A\big(CH_{\bigvee_{i=1}^{r}\big(U_i/\partial U_i\big)}(A), B\big)$$

$$\longrightarrow RHom_A(CH_V(A), B)$$

where the last maps are respectively induced by the map (45), Lemma 4.2 and the map (46). Using the homotopy invariance of Hochschild cochains and unfolding the definition of $\rho_{U_1,\dots,U_r,D}$, we find that the structure map (44) is transfered to the above map (47) under the natural equivalences $RHom_{CH_{\partial U}(A)}^{left}\big(CH_U(A), B\big) \cong CH^{U/\partial U}(A, B)$ (where U is any disk in D^n). Note that when the U_i are cubes in $\mathcal{C}_n(r)$ and D is D^n, the composition of the map (47) with the equivalence

$$\Big(\bigotimes_{i=1}^{r} CH^{S^n}(A, B)\Big) \cong \bigotimes_{i=1}^{r} RHom_{CH_{S^{n-1}}(A,B)}^{left}\big(CH_{D^n}(A), B\big)$$

$$\cong \bigotimes_{i=1}^{r} RHom_{CH_{\partial U_i}(A)}^{left}\big(CH_{U_i}(A), B\big)$$

is the pinching map (26) $pinch_{S^d,r}^*(c) : \big(CH^{S^d}(A, B)\big)^{\otimes r} \longrightarrow CH^{S^d}(A, B)$ (where c is the cube associated to the U_i's). Now, thanks to Theorem 4.12 and the definition of the factorization algebra structure on $U \mapsto Hom_{CH_U(A)}^{\mathcal{E}_n}\big(CH_U(A), B\big)$, we can apply Lemma 6.16 below which implies that the two E_n-algebra structure on $CH^{S^n}(A, B)$ (given by Theorem 4.12 and the one introduced in this proof by the structures maps (47)) are equivalent. $\qquad\square$

Lemma 6.16. *Let $A \in k\text{-}Mod_\infty$ and assume that*

(1) *A has an $\mathcal{C}_n$-algebra structure, i.e., an E_n-algebra structure given by an action of the chains little n-dimensional cube operad.*

(2) *There is a locally constant factorization algebra $\mathcal{A}$ on $\mathbb{R}^n$ (identified with the open unit cube) together with an equivalence $\varphi : \mathcal{A}(\mathbb{R}^n) \xrightarrow{\simeq} A$ (in $k\text{-}Mod_\infty$); which thus induces another E_n-algebra structure on A.*

Assume further that the two structures given by (1) and (2) are compatible *in the following sense: for any configuration of cubes $c \in \mathcal{C}_n(r)$, the following diagram*

$$(48) \qquad \begin{array}{ccc} A^{\otimes r} & \xrightarrow{\mu_c} & A \\ {\scriptstyle \otimes_{i=1}^{r} \varphi \circ \rho_{c_i,\mathbb{R}^n}} \big\uparrow & & \big\uparrow {\scriptstyle \varphi} \\ \mathcal{A}(c_1) \otimes \cdots \otimes \mathcal{A}(c_r) & \xrightarrow{\rho_{c_1,\dots,c_r,\mathbb{R}^n}} & \mathcal{A}(\mathbb{R}^n), \end{array}$$

(where we denote $c_1, \dots, c_r$ the configuration of cubes defined by $c = (c_1, \dots, c_r)$, μ_c is the structure map given by the operadic structure and $\rho_{U_1,\dots,U_r,V}$ the structure maps of the factorization algebra structure), is commutative in $k\text{-}Mod_\infty$.

Then the two E_n-algebras structures on A defined by (1) and (2) are equivalent (in E_n-Alg).

A similar statement holds with E_n-coalgebra structure instead of E_n-algebra structures.

Proof. The statement for coalgebra follows *mutatis mutandis* from the one for algebras; we only prove the last one.

The E_n-algebra structure defined by (1), that is by the action of the little n-dimensional cube operad on A yields a locally factorization algebra $\mathcal{A}'$ on $\mathbb{R}^n$ which is equivalent to the E_n-structure given by (1) and satisfies $\mathcal{A}'(U) \cong \int_U A$ (see Theorem 2.29 and Theorem 2.20). Thus we only have to prove that $\mathcal{A}'$ is equivalent to $\mathcal{A}$ as a factorization algebra on $(0,1)^n$. Let us analyze further the construction of $\mathcal{A}'$. The $\mathcal{C}_n$-action on A gives a structure of $\mathbb{E}_{\mathbb{R}^n}^{\otimes}$-algebra to A, where $\mathbb{E}_{\mathbb{R}^n}^{\otimes}$ is the ∞-operad introduced by Lurie in [L-HA, Section 5.2], that is, the operad whose algebras are precisely those given by Definition 2.4. The canonical map of operad $\mathcal{C}_n \to \mathbb{E}_{\mathbb{R}^n}^{\otimes}$ is an equivalence by the results of Lurie [L-HA, Example 5.2.4.3]. By [L-HA, Theorem 5.2.4.9] (also see § 2.4), the ∞-operad map $N(Disk(\mathbb{R}^n))^{\otimes} \to \mathbb{E}_{\mathbb{R}^n}^{\otimes}$ now yields the locally constant factorization algebra structure on $\mathbb{R}^n$ (denoted $\mathcal{A}'$ above). Let $\mathcal{R}$ be the factorizing basis of $(0,1)^n$ given by the open rectangles and denote $\mathrm{PFac}_{\mathcal{R}}^{lc}$ the category of locally constant $\mathcal{R}$-prefactorization algebras (Definition 2.27). Then, similarly, we have an equivalence $\mathcal{C}_n\text{-Alg} \overset{\cong}{\to} \mathrm{PFac}_{\mathcal{R}}^{lc}$ which fits into the following commutative diagram:

(49)
$$
\begin{array}{ccc}
\mathcal{C}_n\text{-Alg} & \xrightarrow{\;\simeq\;} & \mathrm{PFac}_{\mathcal{R}}^{lc} \\[2pt]
\simeq \uparrow & & \uparrow \\[2pt]
\mathbb{E}_{\mathbb{R}^n}^{\otimes}\text{-Alg} & \xrightarrow{\;\simeq\;} & \mathrm{Fac}_{(0,1)^n}^{lc}
\end{array}
$$

where the right arrow is given by restrictions to the opens belonging to $\mathcal{R}$. In particular, we have natural (with respect to the factorization algebra structure) equivalences $\mathcal{A}'(c) \overset{\cong}{\to} A$ for any open any little rectangle $c \in \mathcal{R}$.

Since the family of open cubes inside $\mathbb{R}^n$ forms a factorization basis of $\mathbb{R}^n$, it is enough, by Proposition 2.28, to check that the factorization algebra structures on $\mathcal{A}'$ and $\mathcal{A}$ are equivalent on the bais $\mathcal{R}$ of open rectangles. From diagram 49, we see that this is precisely the compatibility condition (48) of the Lemma. The result follows. $\qquad\square$

Remark 6.17. It is possible, though more technically involved, to use directly, in the spirit of Section 4.2, the little cube operad $\mathcal{C}_n$ to make $RHom^{\mathcal{E}_n}\big(A, B\big)$ an E_n-algebra. We now sketch how to do this, leaving to the interested reader the task to fill in the many details.

We let again $\mathcal{A}$, $\mathcal{B}$ be the factorization algebras corresponding to A, B. Recall that we have factorizations algebras $\mathcal{A}^{\otimes k}$, $\mathcal{B}^{\otimes k}$ on $\coprod_{i=1}^k D^n$ and similarly for $\mathcal{B}$. Let $c \in \mathcal{C}_n(r)$ be a framed embedding $\coprod_{i=1}^r D^n \hookrightarrow D^n$. Then the little cube c induces a natural (in $\mathcal{A}$ and c) equivalence $\mathcal{A}_{|c^{-1}(D^n)} \cong \mathcal{A}^{\otimes r}$. We can define a map

$$
comp_r(f,c) : RHom^{\mathcal{E}_n}\big(A, B\big)^{\otimes r} \longrightarrow RHom^{\mathcal{E}_n}\big(A, B\big)
$$

similarly to the definition of the structure maps (41). Indeed, we first use c to see the factorization algebras $\mathcal{A}$ and $\mathcal{B}$ on D^n as obtained by gluing together $r + 1$-factorization algebras on D^n. The image $c(\coprod_{i=1}^r D^n)$ has r-open connected components, denoted $D_1, \ldots, D_r$. Choosing small collars $col_1, \ldots col_r$ in the neighborhood of each $D^n \subset \coprod_{i=1}^r D^n$ yield a connected open set $U_\partial := D^n \setminus c\big(\coprod_{i=1}^r (D^n - col_i)\big)$. Since c is an embedding, it induces an identification $\mathcal{A} \cong \mathcal{A}_{|D_i}$ for each $i = 1 \ldots r$. Thus from any family of maps $g_1, \ldots, g_r \in RHom^{\mathcal{E}_n}(\mathcal{A}, \mathcal{B})$, we get induced maps of factorization algebras $g_i : \mathcal{A}_{|D_i} \to \mathcal{B}_{|D_i}$. Further, we also have, by restriction of f to the open set U_c, an induced map $f_c : \mathcal{A}_{|U_c} \to \mathcal{B}_{|U_c}$.

The argument of the proof of Lemma 6.13 apply to show

Lemma 6.18. *The family $(g_1, \ldots, g_r, f_c)$ of maps of factorization algebras glues together to define a map of factorization algebras*

$$comp_r(f, c)\,(g_1, \ldots, g_r) \in Hom(\mathcal{A}, \mathcal{B})$$

Further, on global sections, the induced map $comp_r(f, c)\,(g_1, \ldots, g_r)\big(D^n\big) : A \to B$ is a map of A-E_n-module.

It follows (from the above Lemma 6.13) that we have a well-defined map $(g_1, \ldots, g_r) \mapsto comp_r(f, c)(g_1, \ldots, g_r)(D^n)$, simply denoted by

$$(50) \qquad comp_r(f, c) : RHom^{\mathcal{E}_n}\Big(A, B\Big)^{\otimes r} \longrightarrow RHom^{\mathcal{E}_n}\Big(A, B\Big).$$

Recall that the set of A-E_n-modules homomorphisms is simplicially enriched. Similarly, there are simplicial sets of maps of factorization algebras, see [CG]. Equivalently, we have topological spaces of such maps. Using the fact that the factorization algebras $\mathcal{A}$ and $\mathcal{B}$ are locally constant, one can prove the following

Lemma 6.19. *The map $comp_r(f, c) : RHom^{\mathcal{E}_n}\Big(A, B\Big)^{\otimes r} \longrightarrow RHom^{\mathcal{E}_n}\Big(A, B\Big)$ depends continuously on c.*

The above Lemma 6.19 allows us to consider the maps $comp_r(f, c)$ in families over the operad of little cubes and thus one can let c runs through the operad $\mathcal{C}_n(r)$ so that we get the first part of the following result.

Proposition 6.20. *Let $f : A \to B$ be a map of E_n-algebras.*

(1) *The maps $comp_r(f, c)$ assembles to give a map*

$$comp_r(f) : C_*\big(\mathcal{C}_n(r)\big) \otimes RHom^{\mathcal{E}_n}\Big(A, B\Big)^{\otimes r} \longrightarrow RHom^{\mathcal{E}_n}\Big(A, B\Big)$$

in k-Mod_∞.

(2) *The maps $comp_r(f)$ gives to $RHom^{\mathcal{E}_n}\Big(A, B\Big)$ a natural E_n-algebra structure.*

The proof of the second assertion of this Proposition is essentially the same as the ones of Theorem 6.8 and Theorem 4.12.

6.3. **E_n-Hochschild cohomology as centralizers.** We will now relate the natural E_n-algebra structure of $RHom_A^{\mathcal{E}_n}(A, B)$ (for an E_n-algebra map $f : A \to B$) given in Section 6.2 with the centralizer $\mathfrak{z}(f)$. The following definition is due to Lurie [L-HA, Lu3] (and generalize the notion of center of a category due to Drinfeld).

Definition 6.21. The (derived) centralizer of an E_n-algebra map $f : A \to B$ is the *universal* E_n-algebra $\mathfrak{z}(f)$ endowed with a homomorphism of E_n-algebras $e_{\mathfrak{z}(f)} : A \otimes \mathfrak{z}(f) \to B$ making the following diagram

(51)

$$
\begin{array}{ccc}
 & A \otimes \mathfrak{z}(f) & \\
\scriptstyle{id \otimes 1_{\mathfrak{z}(f)}} \nearrow & & \searrow \scriptstyle{e_{\mathfrak{z}(f)}} \\
A & \xrightarrow{\hspace{1cm} f \hspace{1cm}} & B
\end{array}
$$

commutative in E_n-Alg.

The existence of the derived centralizer $\mathfrak{z}(f)$ of an E_n-algebra map $f : A \to B$ is a *non-trivial* Theorem of Lurie [L-HA, Lu3]. The universal property of the centralizer implies that there are natural maps of E_n-algebras

(52)
$$\mathfrak{z}(\circ) : \mathfrak{z}(f) \otimes \mathfrak{z}(g) \longrightarrow \mathfrak{z}(g \circ f)$$

see [L-HA, Lu3].

The E_n-algebra structure on the $\mathcal{E}_n$-Hochschild cohomology given by Theorem 6.8 gives an explicit description of the centralizer $\mathfrak{z}(f)$ (as an E_n-algebra):

Proposition 6.22. *Let $f : A \to B$ be an E_n-algebra map and endow $HH_{\mathcal{E}_n}(A, B)$ with the E_n-algebra structure given by Theorem 6.8.*

Then the $\mathcal{E}_n$-Hochschild cohomology $HH_{\mathcal{E}_n}(A, B) \cong RHom_A^{\mathcal{E}_n}(A, B)$ is the centralizer $\mathfrak{z}(f)$, i.e., there is a natural equivalence of E_n-algebras $HH_{\mathcal{E}_n}(A, B) \cong \mathfrak{z}(f)$ such that, for any E_n-algebra map $g : B \to C$, the following diagram

$$
\begin{array}{ccc}
RHom_A^{\mathcal{E}_n}(A, B) \otimes RHom_B^{\mathcal{E}_n}(B, C) & \xrightarrow{\cong \otimes \cong} & \mathfrak{z}(f) \otimes \mathfrak{z}(g) \\
\scriptstyle{\circ} \downarrow & & \downarrow \scriptstyle{\mathfrak{z}(\circ)} \\
RHom_A^{\mathcal{E}_n}(A, C) & \xrightarrow{\hspace{1cm} \cong \hspace{1cm}} & \mathfrak{z}(g \circ f)
\end{array}
$$

commutes in E_n-Alg.

Remark 6.23. Note that in the proof of Proposition 6.22, we *do not* assume the existence of centralizers, but actually prove that $HH_{\mathcal{E}_n}(A, B)$ satisfies the universal property of centralizers. In particular the proof of Proposition 6.22 implies the existence of centralizers of any map $f : A \to B$ of E_n-algebras.

We first prove a lemma. Denote $ev : A \otimes RHom_A^{\mathcal{E}_n}(A, B) \to B$ the (derived) evaluation map $(a, f) \mapsto f(a)$.

Lemma 6.24. *The evaluation map $ev : A \otimes RHom_A^{\mathcal{E}_n}(A, B) \to B$ is an E_n-algebra morphism. Further, the following diagram*

$$
\begin{array}{ccc}
 & A \otimes RHom_A^{\mathcal{E}_n}(A, B) & \\
\scriptstyle{id \otimes 1} \nearrow & & \searrow \scriptstyle{ev} \\
A & \xrightarrow{\hspace{1cm} f \hspace{1cm}} & B
\end{array}
$$

is commutative in E_n-Alg.

Proof. There are canonical equivalences of E_n-algebras $RHom_k^{\mathcal{E}_n}(k, A) \cong A$ and $RHom_k^{\mathcal{E}_n}(k, B) \cong B$ (see Example 6.11). Thus, the fact that ev is a map of E_n-algebras follows from statement (3) in Theorem 6.8. Further, the same Theorem

implies that the unit of $RHom_A^{\mathcal{E}_n}(A, B)$ is $f : A \to B$. It follows that $ev\circ(id\otimes 1) = f$ which proves the Lemma. $\square$

Proof of Proposition 6.22. By Lemma 6.24, we already know that $HH_{\mathcal{E}_n}^\bullet(A, B) \cong RHom_A^{\mathcal{E}_n}(A, B)$ fits into a commutative diagram similar to diagram (53) below. We have to prove that for any E_n-algebra $\mathfrak{z}$, endowed with a E_n-algebra map $\phi : A \otimes \mathfrak{z} \to B$ fitting in a commutative diagram

(53)

$$
\begin{array}{ccc}
 & A \otimes \mathfrak{z} & \\
{\scriptstyle id\otimes 1_{\mathfrak{z}}}\nearrow & & \searrow{\scriptstyle \phi} \\
A & \xrightarrow{\quad f \quad} & B,
\end{array}
$$

there exists an E_n-algebra map $\mathfrak{z} \to RHom_A^{\mathcal{E}_n}$ which makes $A \otimes \mathfrak{z} \xrightarrow{\phi} B$ factor through $A \otimes RHom_A^{\mathcal{E}_n}(A, B) \xrightarrow{ev} B$.

Let $\theta_\phi : \mathfrak{z} \to RHom(A, B)$ be the map associated to $\phi : A \otimes \mathfrak{z} \to B$ under the (derived) adjunction $RHom(A \otimes \mathfrak{z}, B) \cong RHom(\mathfrak{z}, RHom(A, B))$ (in $k\text{-Mod}_\infty$).

We now prove that θ_ϕ takes values in $RHom_A^{\mathcal{E}_n}(A, B)$. We use again the factorization algebra characterization of E_n-algebras. Let $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{Z}$ be the locally constant factorization algebras associated to A, B and $\mathfrak{z}$. For any open sub-disk $D \hookrightarrow D^n$, we get the induced map[33]

$$
\phi : \big(\mathcal{A} \otimes \mathcal{Z}\big)(D) \cong \int_D A \otimes \int_D \mathfrak{z} \xrightarrow{\int_D \phi} \int_D B \cong \mathcal{B}(D)
$$

and its (derived) adjoint $\theta_\phi : \mathcal{Z}(D) \longrightarrow RHom(\mathcal{A}(D), \mathcal{B}(D))$. We are left to check that this last map is compatible with the factorization algebra structures (describing the A-E_n-module structure of A and B). Let $U_0, U_1, \ldots, U_r$ be pairwise disjoints open disks included in a bigger disk V, where we assume that U_0 contains the base point of D^n. Also we use the same notation

$$
\rho_{U_0,\ldots,U_r,V} : \mathcal{F}(U_0) \otimes \cdots \otimes \mathcal{F}(U_r) \longrightarrow \mathcal{F}(V)
$$

for the associated structure maps of any one of the factorization algebras $\mathcal{F} = \mathcal{A}$, $\mathcal{B}$ or $\mathcal{Z}$ on D^n. Since $\phi : A \otimes \mathfrak{z} \to B$ is a map of E_n-algebras, for any $a_i \in \mathcal{A}(U_i)$ $(i = 1 \ldots r)$, $x \in \mathcal{A}(U_0)$ and $z \in \mathfrak{z}(U_0)$, we have

$$
\begin{aligned}
\phi\Big(\rho_{U_0,\ldots,U_r,V}\big(x, a_1, \ldots, a_r\big) \otimes \rho_{U_0,V}\big(z\big)\Big) &= \phi\Big(\rho_{U_0,\ldots,U_r,V}\big(x \otimes z, a_1 \otimes 1_{\mathfrak{z}}, \ldots, a_r \otimes 1_{\mathfrak{z}}\big)\Big) \\
&= \rho_{U_0,\ldots,U_r,V}\Big(\phi(x \otimes z), \phi(a_1 \otimes 1_{\mathfrak{z}}), \ldots \\
&\qquad\qquad\qquad \ldots, \phi(a_r \otimes 1_{\mathfrak{z}})\Big) \\
&= \rho_{U_0,\ldots,U_r,V}\big(\phi(x \otimes z), f(a_1), \ldots, f(a_r)\big)
\end{aligned}
$$

where the last identity follows from the commutativity of diagram (53). Note that the map $z \mapsto \rho_{U_0,V}\big(z\big)$ is an equivalence (since $\mathcal{Z}$ is locally constant). Since the A-E_n-module structure on B is given by f, the above string of equalities ensures that

---

[33]we make, for simplicity, an abuse of notation still denoting by ϕ the induced map and similarly with θ_ϕ below

θ_ϕ is a map from $\mathfrak{z}$ to $RHom_A^{\mathcal{E}_n}(A, B)$. In particular, the map $\theta_\phi : \mathfrak{z} \to RHom(A, B)$ factors as

$$\mathfrak{z} \xrightarrow{\tilde{\theta}_\phi} RHom_A^{\mathcal{E}_n}(A, B) \cong RHom_{\int_{S^{n-1}} A}^{left}\Big(\int_{D^n} A, \int_{D^n} B \Big) \hookrightarrow RHom(A, B).$$

To finish the proof of Proposition 6.22, we need to check that $\tilde{\theta}_\phi : \mathfrak{z} \to RHom_A^{\mathcal{E}_n}(A, B)$ is a map of E_n-algebras. Recall that there are equivalences $\mathfrak{z} \cong RHom_k^{\mathcal{E}_n}(k, \mathfrak{z})$, $\mathfrak{z} \cong k \cong \mathfrak{z}$ of E_n-algebras (see Example 6.11). By definition of the (derived) adjunction, $\tilde{\theta}_\phi : \mathfrak{z} \to RHom_A^{\mathcal{E}_n}(A, B)$ factors as the composition

(54)

$$\mathfrak{z} \cong k \otimes \mathfrak{z} \xrightarrow{1_{RHom_A^{\mathcal{E}_n}(A,A)} \otimes id} RHom_A^{\mathcal{E}_n}(A, A) \otimes \mathfrak{z} \cong RHom_A^{\mathcal{E}_n}(A, A) \otimes RHom_k^{\mathcal{E}_n}(k, \mathfrak{z})$$
$$\longrightarrow RHom_A^{\mathcal{E}_n}(A, A \otimes \mathfrak{z}) \xrightarrow{\phi_*} RHom_A^{\mathcal{E}_n}(A, B).$$

By Theorem 6.8.(2) and (4), the last two maps are an E_n-algebra Homomorphisms. Thus the composition (54) is a composition of E_n-algebras maps hence $\tilde{\theta}_\phi : \mathfrak{z} \to RHom_A^{\mathcal{E}_n}(A, B)$ itself is a map of E_n-algebras.

Further, by definition of θ_ϕ, the identity

$$ev \circ \big(id_A \otimes \theta_\phi \big) = \phi$$

holds. Hence we eventually get a commutative diagram

in E_n-Alg.

It remains to prove the uniqueness of the map $\mathfrak{z} \to RHom_A^{\mathcal{E}_n}(A, B)$ inducing such a commutative diagram. Thus assume that $\alpha : \mathfrak{z} \to RHom_A^{\mathcal{E}_n}(A, B)$ is a map of E_n-algebras such that the following diagram

(55)

is commutative in E_n-Alg. Note that the composition

$$(56) \quad RHom_A^{\mathcal{E}_n}(A, B) \cong RHom_k^{\mathcal{E}_n}\left(k, RHom_A^{\mathcal{E}_n}(A, B)\right)$$

$$\xrightarrow{1_{RHom_A^{\mathcal{E}_n}(A,A)} \otimes id} RHom_A^{\mathcal{E}_n}(A, A) \otimes \left(k, RHom_A^{\mathcal{E}_n}(A, B)\right)$$

$$\longrightarrow RHom_A^{\mathcal{E}_n}\left(A, A \otimes RHom_A^{\mathcal{E}_n}(A, B)\right)$$

$$\xrightarrow{ev_*} RHom_A^{\mathcal{E}_n}(A, B)$$

is the identy map. From the commutativity of Diagram (55), we get the following commutative diagram
(57)

in E_n-Alg. The composition of the right vertical maps in Diagram (57) is the composition (56), hence is the identity, and the upper map is $\alpha : \mathfrak{z} \to RHom_A^{\mathcal{E}_n}(A, B)$. It follows that the map α is equivalent to the map (54) hence to $\tilde{\theta}_\phi$. This gives the uniqueness statement and the Proposition will follow once we proved the diagram depicted in Proposition 6.22 is commutative. The latter is an immediate consequence of the universal property of the centralizers (and thus of $HH_A^{\mathcal{E}_n}(A, B)$) and of Theorem 6.8.(3). $\qquad\square$

Example 6.25 (E_n-Koszul duality). Assume $B = k$ so that $f : A \to k$ is an augmentation. Then by Proposition 6.22 and [L-HA, Example 6.1.4.14] and [Lu-MP, Remark 7.13] there is an equivalence of E_n-algebras

$$HH_{\mathcal{E}_n}(A, k) \cong RHom(Bar^{(n)}(A), k)$$

where $Bar^{(n)}(A)$ is the E_n-coalgebra given by the iterated Bar construction on A, that is, the (derived) *Koszul dual* of A. Thus Theorem 6.8 gives an explicit description of the E_n-algebra structure on the dual of $Bar^{(n)}(A)$. See Section 8 for a more detailed description.

Combining Proposition 6.22 and Proposition 6.15, we get

Corollary 6.26. *let $f : A \to B$ be a map of E_∞-algebras. Then the Hochschild cochains $CH^{S^n}(A, B)$ over the n-sphere (endowed with its E_n-algebra structure given by Theorem 4.12) is the centralizer $\mathfrak{z}(f)$ of f viewed as a map of E_n-algebras (by restriction).*

Remark 6.27. Assume $f : A \to B$ and $g : B \to C$ are maps of CDGA's. Then by the above Corollary 6.26 or Proposition 6.15, there is a composition

$$(58) \qquad CH^{S^n}(A, B) \otimes CH^{S^n}(B, C) \xrightarrow{\circ} CH^{S^n}(A, C)$$

(which is a map of E_n-algebras) induced by the natural equivalence $CH^{S^n}(A, B) \cong$ $RHom^{left}_{CH_{S^{n-1}}(A)}(CH_{D^n}(A), B)$ and (derived) compositions of homomorphisms. In the setting of CDGA's, this composition can be represented in an easy way as follows. Let $I_\bullet$ be the standard simplicial model of the interval ([G1, GTZ]); its boundary $\partial I_\bullet^n$ is a simplicial model for S^{n-1}. Then the map (58) is represented by the usual composition (of left dg-modules)

$$Hom^{left}_{CH_{\partial I_\bullet^n}(A)}\left(CH_{I_\bullet^n}(A), CH_{I_\bullet^n}(B)\right) \otimes Hom^{left}_{CH_{\partial I_\bullet^n}(B)}\left(CH_{I_\bullet^n}(B), CH_{I_\bullet^n}(C)\right)$$
$$\xrightarrow{\circ} Hom^{left}_{CH_{\partial I_\bullet^n}(A)}\left(CH_{I_\bullet^n}(A), CH_{I_\bullet^n}(C)\right)$$

since $CH_{I_\bullet^n}(A)$ is a (semi-)free $CH_{\partial I_\bullet^n}(A)$-algebra.

6.4. **The higher Deligne conjecture.** In this section we deal with (some of) the solutions of the higher Deligne conjecture. That is we specialized the results of the previous sections 6 and 4.2 to the case $A = B$ and $f = id$.

By Theorem 6.8 above, the composition of morphisms of A-E_n-modules

$$(59) \qquad RHom^{\mathcal{E}_n}_A(A, A) \otimes RHom^{\mathcal{E}_n}_A(A, A) \xrightarrow{\circ} RHom^{\mathcal{E}_n}_A(A, A)$$

is a homomorphism of E_n-algebras (with unit given by the identity map $id : A \to A$). The composition of morphisms is further (homotopy) associative and unital (with unit id); thus $RHom^{\mathcal{E}_n}_A(A, A)$ is actually an E_1-algebra in the ∞-category E_n-Alg.

By the ∞-category version of Dunn Theorem [Du, L-HA, Lu3] or see Theorem 2.30, there is an equivalence of $(\infty, 1)$-categories $E_1 - Alg\left(E_n - Alg\right) \cong E_{n+1} - Alg$. Thus the multiplication (59) lift the E_n-algebra structure of $HH^\bullet_{\mathcal{E}_n}(A, A) \cong RHom^{\mathcal{E}_n}_A(A, A)$ to an E_{n+1}-algebra structure.

In particular we just proved the first part of the following result, which has already been given by Francis [F1] (and Lurie [L-HA, Lu3]).

Theorem 6.28. (Higher Deligne Conjecture)

(1) *Let A be an E_n-algebra. There is a natural E_{n+1}-algebra structure on $HH^\bullet_{\mathcal{E}_n}(A, A)$ with underlying E_n-algebra structure given by Theorem 6.8[34].*

(2) *Let now A be an E_∞-algebra. Then there is a natural E_{n+1}-algebra structure on $CH^{S^n}(A, A)$ whose underlying E_n-algebra structure is the one given by Theorem 4.12. In particular, the underlying E_1-algebra structure is given by the standard cup-product (see Corollary 4.6 and Example 4.7).*

(3) *For A an E_∞-algebra, the two E_{n+1}-structures given by statements (1) and (2) are equivalent.*

Proof. We have already proved the first claim. Note that the underlying E_n-algebra structure of an E_{n+1}-algebra is induced by the pushforward along the canonical projection $\mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$, see Theorem 2.30. By Proposition 6.2, $CH^{S^n}(A, A)$ also inherits a structure of E_{n+1}-algebra whose underlying E_n-algebra is the same as the one given by Theorem 4.12 thanks to Proposition 6.15. This proves both claims (2) and (3). $\square$

---

[34]where we take $B = A$ and $f = id$

Example 6.29. In the case $n = 1$, Theorem 6.28 recovers the original Deligne conjecture asserting the existence of a natural E_2-algebra structure on the Hochschild cochains lifting the associative algebra structure induced by the cup-product. It can be proved that this E_2-algebra structure induces the usual Gerstenhaber algebra structure (from [Ge]) on the Hochschild cohomology groups.

Remark 6.30. Francis [F1] has given a different solution to the higher Deligne conjecture. His solution is directly and explicitly related to the degree n Lie algebra structure on $HH_{\mathcal{E}_n}(A, A)$. However, the underlying cup-product (*i.e.* E_1-algebra structure) is more mysterious. This is in contrast to the solution given by Theorem 6.28. This latter solution is, by definition, the same as the one of Lurie [L-HA]. It would therefore be very interesting and useful to relate Francis' construction to ours. Note that the explicit knowledge of the cup-product is useful to us to relate this construction to the higher string topology operations, see § 7.

6.5. **Explicit computations via higher formality.** For the remaining of this section, we work over *a characteristic zero field* and we let A be a commutative differential graded algebra. In that case, we can use the higher formality Theorem [To, CW] mentioned first in [PTVV] to compute more explicitly the E_{n+1}-algebra structure on the derived center $HH^\bullet_{\mathcal{E}_n}(A, A)$. We will start by identifying our E_{n+1}-algebra structure with the one obtained very recently by Toën [To].

6.5.1. *P_n-algebras and Toën Brane operations.* In characteristic zero, the operad $\mathbb{E}_n$ is formal when $n \geq 2$ ([LV]) so that one can choose[35] an equivalence of ∞-operads $\mathbb{E}_n \to \mathbb{P}_n$ where $\mathbb{P}_n$ is the (∞-operad associated to the) operad governing (homotopy) P_n-algebras (in particular $\mathbb{P}_n \cong H_\bullet(\mathbb{E}_n)$ for $n \geq 2$). More precisely, we have the differential graded operad $\mathbb{P}_n$ which is the (cofibrant) minimal resolution of the operad $P_n \cong H_\bullet(\mathbb{P}_n)$ of (strict, differential graded) P_n-algebras. These two operads yields (using the standard nerve functor from operads to ∞-operads) equivalent ∞-operads (and dendroidal sets), which we still denote $\mathbb{P}_n$.

By a P_n-algebra[36] we mean a differential graded commutative unital algebra $(B, d, \cdot)$ equipped with a (homological) degree $n-1$ bracket which makes the iterated suspension $A[1 - n]$ a differential graded Lie algebra. The bracket and product are further require to satisfy the graded Leibniz identity $[a \cdot b, c] = \pm a \cdot [b, c] + \pm b \cdot [a, c]$. In other words P_2-algebras are the same as (differential graded) Gerstenhaber algebras.

In particular, if B is a P_{n+1}-algebra, then $B[-n]$ is a differential graded Lie algebra. By the above choice of formality maps $\mathbb{E}_n \to \mathbb{P}_n$ of operads, then for any E_{n+1}-algebra H, $H[-n]$ inherits an (homotopy) dg-Lie algebra structure as well.

Slightly after a first draft of our paper was written, Toën used the machinery of ∞-operad of configuration type to prove the following result.

Theorem 6.31 (Toën [To]). *Let $\mathfrak{X}$ be a derived stack and $n \geq 2$. There is a canonical equivalence of (homotopy) dg-Lie algebras*

$$\mathbb{R}\Gamma\big(\mathfrak{X}, \mathrm{Sym}_{\mathcal{O}_\mathfrak{X}}\big(\mathbb{T}_\mathfrak{X}[n]\big)\big)[-n] \cong HH^\bullet_{\mathcal{E}_n}(\mathcal{O}_\mathfrak{X}, \mathcal{O}_\mathfrak{X})[-n].$$

Here both sides are given a Lie algebra structure induced by a canonical P_n and E_n-algebras structures constructed in [To, § 5].

---

[35]there are however many possible choices, for instance see [Ta3]

[36]which are sometimes called n-algebras [GeJo, G1] or e_n-algebras [Ta3, CW] in the literature

We are only interested in theorem 6.31 in the case when $\mathfrak{X} = \mathbb{R}\mathrm{Spec}(A)$ is affine, given by a *(differential graded) commutative algebra A*. In that case, the tangent complex $\mathbb{T}_\mathfrak{X}$ is equivalent to $\mathbb{R}\mathrm{Der}(A, A)$ the right derived functor of derivations of A. Hence, theorem 6.31 provides a canonical equivalence of *(homotopy) dg-Lie algebras*:

$$(60) \qquad \mathrm{Sym}_A\big(\mathbb{R}\mathrm{Der}(A, A)[n]\big)[-n] \cong HH^\bullet_{\mathcal{E}_n}(\mathcal{O}_{\mathbb{R}\mathrm{Spec}(A)}, \mathcal{O}_{\mathbb{R}\mathrm{Spec}(A)})[-n].$$

In fact, Toën deduced the above equivalence (60) from *an equivalence of objects in* E_1-*Alg*(E_n-*Alg*) which is a consequence of the main Theorem of [To]. Note that the left hand side $\mathrm{Sym}_A\big(\mathbb{R}\mathrm{Der}(A, A)[n]\big)$ of (60) is the $\mathbb{P}_n$-branes cohomology of $\mathbb{R}\mathrm{Spec}(A)$ while the right hand side $HH^\bullet_{\mathcal{E}_n}(\mathcal{O}_{\mathbb{R}\mathrm{Spec}(A)}, \mathcal{O}_{\mathbb{R}\mathrm{Spec}(A)})$ is its $\mathbb{E}_n$-branes cohomology.

It is essentially immediate from the definition[37] that, *as a cochain complex,* $HH^\bullet_{\mathcal{E}_n}(\mathcal{O}_{\mathbb{R}\mathrm{Spec}(A)}, \mathcal{O}_{\mathbb{R}\mathrm{Spec}(A)})$ is equivalent to $HH^\bullet_{\mathcal{E}_n}(A, A)$, the E_n-center of A (Definition 6.1 and Corollary 6.26) . We prove below that the equivalence is actually an equivalence of E_{n+1}-algebras between the right hand side of (60) and our solution to Deligne conjecture from § 6.3.

To do so, for $n \geq 2$, we first choose equivalences of ∞-operad $\mathcal{F}_n : \mathbb{E}_n \xrightarrow{\simeq} \mathbb{P}_n$.

Denote $\mathbb{O} \odot \mathbb{P}$ the tensor product of two ∞-operads $\mathbb{O}$ and $\mathbb{P}$. It is an ∞-operad governing $\mathbb{O}$-algebras in $\mathbb{P}$-algebras. If one uses the model given by dendroidal sets for ∞-operads, the tensor product is represented by the (derived) tensor product of dendroidal sets (which forms a symmetric monoidal model category) of [CiMo, CiMo2, CiMo3].

Recall also, that, by Dunn Theorem 2.30 (see [L-HA]), we have an equivalence $D_{n+1} : \mathbb{E}_1 \odot \mathbb{E}_n \xrightarrow{\simeq} \mathbb{E}_{n+1}$ of ∞-operads. We sum up these two facts in the

Proposition 6.32. *Let $n \geq 2$. There are equivalences of ∞-operads*

$$\mathbb{E}_1 \odot \mathbb{P}_n \xleftarrow[\simeq]{id \odot \mathcal{F}_n} \mathbb{E}_1 \odot \mathbb{E}_n \xrightarrow[\simeq]{D_{n+1}} \mathbb{E}_{n+1} \xrightarrow[\simeq]{\mathcal{F}_{n+1}} \mathbb{P}_{n+1}$$

Here $\mathbb{E}_1 \odot \mathbb{P}_n$ (resp. $\mathbb{E}_1 \odot \mathbb{E}_n$) are the ∞-operad governing E_1-algebras in the (symmetric monoidal ∞-)category of P_n-algebras (resp. E_n-algebras).

In particular, we thus get equivalences $\mathbb{E}_1 \odot \mathbb{P}_n \cong \mathbb{P}_{n+1}$ (for $n \geq 2$) fitting into a commutative diagram of equivalences of ∞-operads:

(61)

$$
\begin{array}{ccccccccc}
\mathbb{E}_1 \odot \cdots \odot \mathbb{E}_1 \odot \mathbb{E}_2 & \xrightarrow{\simeq} & \cdots & \longrightarrow & \mathbb{E}_1 \odot \mathbb{E}_1 \odot \mathbb{E}_{n-1} & \xrightarrow{\simeq} & \mathbb{E}_1 \odot \mathbb{E}_n & \xrightarrow{\simeq} & \mathbb{E}_{n+1} \\
\downarrow{\simeq} & & \downarrow{\simeq} & & \downarrow{\simeq} & & \downarrow{\simeq} & & \downarrow{\simeq} \\
\mathbb{E}_1 \odot \cdots \odot \mathbb{E}_1 \odot \mathbb{P}_2 & \xrightarrow{\simeq} & \cdots & \xrightarrow{\simeq} & \mathbb{E}_1 \odot \mathbb{E}_1 \odot \mathbb{P}_{n-1} & \xrightarrow{\simeq} & \mathbb{E}_1 \odot \mathbb{P}_n & \xrightarrow{\simeq} & \mathbb{P}_{n+1}
\end{array}
$$

where the vertical pointing down arrows are induced by the formality maps $\mathcal{F}_\rangle$ and the horizontal upper arrows by iteration of Dunn Theorem.

We can identify the right hand side of (60) with the structure given by the Deligne conjecture (Theorem 6.28):

Proposition 6.33. *One has a canonical equivalence of $\mathbb{E}_1 \odot \mathbb{E}_n$-algebras*

$$HH^\bullet_{\mathcal{E}_n}(\mathcal{O}_{\mathbb{R}\mathrm{Spec}(A)}, \mathcal{O}_{\mathbb{R}\mathrm{Spec}(A)}) \cong CH^{S^n}(A, A)$$

---

[37]see [To, §5] or the proof of Proposition 6.33 below

where the right hand side is given the structure given by Theorem 6.28 and the left hand side is given the structure of [To, § 5]*.*

Proof. Toën has proved in [To] (in particular Corollary 5.1 in *loc. cit.*) that the $\mathbb{E}_n$-branes cohomology $HH^\bullet_{\mathcal{E}_n}(\mathcal{O}_{\mathbb{R}\mathrm{Spec}(A)}, \mathcal{O}_{\mathbb{R}\mathrm{Spec}(A)})$ of $\mathbb{R}\mathrm{Spec}(A)$ is an $E_1 \circledcirc E_n$-algebra. It is given, by definition, by

$$RHom_{D\left(\mathcal{L}^{n-1}(\mathbb{R}\mathrm{Spec}(A))\right)}\left(\mathcal{O}_{\mathbb{R}\mathrm{Spec}(A)}, \mathcal{O}_{\mathbb{R}\mathrm{Spec}(A)}\right)$$

where $\mathcal{L}^{n-1}(\mathbb{R}\mathrm{Spec}(A))$ is the iterated (derived) loop stack $\mathbb{R}Map(S^{k-1}, \mathbb{R}\mathrm{Spec}(A))$. By Corollary 3.15, we get that the derived category $D\left(\mathcal{L}_f^{n-1}(\mathbb{R}\mathrm{Spec}(A))\right)$ is equivalent to $CH_{S^{k-1}}(A)$-LMod. Hence, we get an equivalence of E_1-algebras (for the structure given by composition of endomorphisms):

$$(62) \quad RHom_{D\left(\mathcal{L}^{n-1}(\mathbb{R}\mathrm{Spec}(A))\right)}\left(\mathcal{O}_{\mathbb{R}\mathrm{Spec}(A)}, \mathcal{O}_{\mathbb{R}\mathrm{Spec}(A)}\right) \cong RHom^{left}_{CH_{S^{n-1}}(A)}(A, A).$$

Toën has proved that the left hand side has an E_n-algebra structure given by a natural map of ∞-operads $\left(\mathcal{C}_n(r)\right)_r \to End\left(D\left(\mathcal{L}^{n-1}(\mathbb{R}\mathrm{Spec}(A))\right)\right)$. Under the above equivalence, this action of the little cubes is seen to be transfered to the map (47) (for any family of little cubes U_i inside the unit cube D) defined in the proof of Proposition 6.15. The proof of this Proposition also shows that this E_n-algebra structure on $RHom^{left}_{CH_{S^{n-1}}(A)}(A, A)$ is equivalent to the one on $CH^{S^n}(A, A)$ given by Theorem 4.12 in a natural way. It follows that the equivalence (62) is an equivalence of $E_1 \circledcirc E_n$-algebras. □

Corollary 6.34. *For any choice of formality equivalence as in Proposition 6.33, one has canonical equivalences of E_{n+1}-algebras (and homotopy P_{n+1}-algebras as well)*

$$HH^\bullet_{\mathcal{E}_n}(\mathcal{O}_{\mathbb{R}\mathrm{Spec}(A)}, \mathcal{O}_{\mathbb{R}\mathrm{Spec}(A)}) \cong \mathrm{Sym}_A\left(\mathbb{R}\mathrm{Der}(A, A)[n]\right) \cong CH^{S^n}(A, A) \cong HH^\bullet_{\mathcal{E}_n}(A, A)$$

where the two right hand sides are given the structure given by Theorem 6.28 and the left hand sides are given the structure of [To, § 5]*.*

Proof. The choice of formality provides an equivalence $(id \circledcirc \mathcal{F}_n)^* : \mathbb{E}_1 \circledcirc \mathbb{P}_n\text{-Alg} \xrightarrow{\cong} \mathbb{E}_1 \circledcirc \mathbb{E}_n\text{-Alg}$. Note that that the map of (∞-)operad $\mathbb{E}_n \xrightarrow{\mathcal{F}_n} \mathbb{P}_n \to Comm$ is the canonical one. Further there is a canonical equivalence $\mathbb{E}_1 \otimes Comm \xrightarrow{\simeq} Comm$ and the following diagram of ∞-operads

$$
\begin{array}{ccccccc}
\mathbb{E}_1 \circledcirc \mathbb{P}_n & \xleftarrow[\simeq]{id \circledcirc \mathcal{F}_n} & \mathbb{E}_1 \circledcirc \mathbb{E}_n & \xrightarrow[\simeq]{D_{n+1}} & \mathbb{E}_{n+1} & \xrightarrow[\simeq]{\mathcal{F}_{n+1}} & \mathbb{P}_{n+1} \\
& \searrow & \downarrow & & \downarrow & \swarrow & \\
& & \mathbb{E}_1 \otimes Comm & \xrightarrow{\simeq} & Comm & &
\end{array}
$$

is commutative.

Since A is a CDGA seen as both a P_n-algebra with trivial bracket and an E_n-algebra through the above maps $\mathbb{P}_n \to Comm$ and $\mathbb{E}_n \to Comm$, it follows that the space of P_n-branes[38] and the space of $\mathbb{E}_n$-branes[39] of A (in the sense of [To]) are equivalent as $\mathbb{E}_1 \circledcirc \mathbb{E}_n$-algebras.

---

[38] which is $\mathrm{Sym}_A\left(\mathbb{R}\mathrm{Der}(A, A)[n]\right)$

[39] which is $HH^\bullet_{\mathcal{E}_n}(\mathcal{O}_{\mathbb{R}\mathrm{Spec}(A)}, \mathcal{O}_{\mathbb{R}\mathrm{Spec}(A)})$

Now, in view of Proposition 6.32 and Theorem 6.28, the result is a corollary of Proposition 6.33. □

6.5.2. *Higher formality and Tamarkin homotopy P_{n+1}-structure.* By Corollary 6.34, $\mathrm{Sym}_A\big(\mathbb{R}\mathrm{Der}(A, A)[n]\big)$ inherits an homotopy P_{n+1}-algebra structure induced by its interpration (due to [To]) as $\mathbb{P}_n$-branes cohomology.

There is also a canonical (strict) P_{n+1}-algebra structure on $\mathrm{Sym}_A\big(\mathbb{R}\mathrm{Der}(A, A)[n]\big)$. Indeed, there is a Lie algebra structure on $\mathrm{Sym}_A\big(\mathbb{R}\mathrm{Der}(A, A)[n]\big)[-n]$ given by the Schouten bracket. More precisely, $\mathbb{R}\mathrm{Der}(A, A)$ has a canonical differential graded Lie algebra structure[40] such that the canonical map $\mathrm{Der}(A, A) \to \mathbb{R}\mathrm{Der}(A, A)$ is a map of (dg-)Lie algebras.

Then, $\mathrm{Sym}_A\big(\mathbb{R}\mathrm{Der}(A, A)[n]\big)$ is made into a P_{n+1}-algebra whose underlying CDGA structure is given by the (graded) symmetric algebra construction on the (dg-) A-module $\mathbb{R}\mathrm{Der}(A, A)[n]$. There is a unique extension of the Lie bracket on $(\mathbb{R}\mathrm{Der}(A, A)[n])[-n] = \mathbb{R}\mathrm{Der}(A, A)[n]$ satisfying the Leibniz rule, which defines the P_{n+1}-algebra structure. This (strict) P_{n+1}-structure induces canonically a $\mathbb{P}_{n+1}$-structure (*i.e.* an homotopy P_{n+1}-structure) on $\mathrm{Sym}_A\big(\mathbb{R}\mathrm{Der}(A, A)[n]\big)$ and thus, given any choice of a formality map, an E_{n+1}-algebra structure as well.

Associated to any P_n-algebra V, one can define its cohomology complex $HH_{\mathbb{P}_n}^\bullet(V, V)$ which by a result of Tamarkin [Ta2] has a canonical homotopy P_{n+1}-algebra structure.

Calaque and Willwacher have recently proved the following higher formality relating the P_{n+1}-structure of $\mathrm{Sym}_A\big(\mathbb{R}\mathrm{Der}(A, A)[n]\big)$ and the P_{n+1}-structure of $HH_{\mathbb{P}_n}^\bullet(A, A)$ for a CDGA A, seen as a P_n-algebra with trivial bracket.

Theorem 6.35 (Calaque Willwacher [CW])**.** *Let A be a differential graded commutative algebra over a characteristic zero field. There is a canonical equivalence of $\mathbb{P}_{n+1}$-algebras:*

$$(63) \qquad \mathrm{Sym}_A\big(\mathbb{R}\mathrm{Der}(A, A)[n]\big) \cong HH_{\mathbb{P}_n}^\bullet(A, A)$$

where the right hand side is endowed with the (homotopy) P_{n+1}-structure constructed by Tamarkin in [Ta2].

The left hand side of (63) has a very explicit P_{n+1}-structure. The same complex has another homotopy P_{n+1}-structure given by Corollary 6.34, that is, by our solution to the higher Deligne conjecture for $n \geq 2$. In order to prove that these two structures are actually the same, now, we only need to check that Tamarkin structure on $HH_{\mathbb{P}_n}^\bullet(A, A)$ is equivalent to the one given by the center (and thus Toen's one as well).

The cochain complex $HH_{\mathbb{P}_n}^\bullet(A, A)$ is defined in [Ta2, §2] as follows. We denote $\mathbb{P}_n^\vee(A)$ the (coproartinian) cofree P_n-coalgebra on $A[-n]$ equipped with the differential $\partial_{\mathbb{P}_n^\vee(A)}$ corresponding to the (homotopy) P_n-algebra structure of A (this is a coderivation of $\mathbb{P}_n^\vee(A)$). Let $\mathrm{coDer}(\mathbb{P}_n^\vee(A), \mathbb{P}_n^\vee(A))$ be the vector space of coderivations of $\mathbb{P}_n^\vee(A)$. Then the cochain complex $HH_{\mathbb{P}_n}^\bullet(A, A)$ is $\mathrm{coDer}(\mathbb{P}_n^\vee(A), \mathbb{P}_n^\vee(A))$ equipped with the differential $[\partial_{\mathbb{P}_n^\vee(A)}, -]$ obtained as the bracket of a coderivation with $\partial_{\mathbb{P}_n^\vee(A)}$.

---

[40]given by the standard Lie algebra structure on the derivations $\mathrm{Der}(P_A, P_A)$ where $P_A \to A$ is a resolution of A by semi-free CDGAs

Lemma 6.36. *Let A be a differential graded commutative algebra over a characteristic zero field. There is an natural equivalence of P_{n+1}-algebras*

$$HH^\bullet_{\mathbb{P}_n}(A, A) \cong CH^{S^n}(A, A)$$

where the right hand side is endowed with the structure given by Theorem 6.28 and the left hand side is endowed with the one constructed by Tamarkin in [Ta2].

Proof. Since $\mathbb{P}_n^\vee(A)$ is a coproartinian cofree coalgebra, there is an isomorphism $\mathrm{coDer}(\mathbb{P}_n^\vee(A), \mathbb{P}_n^\vee(A)) \cong Hom(\mathbb{P}_n^\vee(A), A[-n])$ from which one deduced an isomorphism of cochain complexes $HH^\bullet_{\mathbb{P}_n}(A, A) \cong Hom(\mathbb{P}_n^\vee(A), A)[-n]$ where the right hand side is endowed with the inner differential of $\mathbb{P}_n^\vee(A)$ twisted by the canonical map $\mathbb{P}_n^\vee(A) \to A[-n]$, see [Ta2, §4]. Further, in *loc. cit.*, Tamarkin proved that $Hom(\mathbb{P}_n^\vee(A), A)[-n]$ is an homotopy P_n-algebra so that $\mathbb{P}_n^\vee\Big(Hom(\mathbb{P}_n^\vee(A), A)[-n]\Big)$ is a n-bialgebra in the sense of [Ta2, §4], that is an E_1-algebra in the category of homotopy P_n-algebras. The latter chain complex is also denoted $\underline{Hom}^{Id}(A, A)$ in [Ta2, §3].

More generally, Tamarkin proved that, associated to any P_n-algebra morphism $\phi : A \to B$, one obtains a similar way an homotopy P_n-algebra structure on $Hom(\mathbb{P}_n^\vee(A), B)[-n]$, with underlying differential given by the inner differential of $\mathbb{P}_n^\vee(A)$ and B and twisted by ϕ. This structure is equivalent to a differential on the cofree coalgebra $\mathbb{P}_n^\vee\Big(Hom(\mathbb{P}_n^\vee(A), B)[-n]\Big)$ which corepresents the canonical functor $F^\phi_{A,B} : d - coart^0 \to Sets$ of moduli problems for P_n-algebras, see [Ta2, §2 and 3]. It is also denoted $\underline{Hom}^\phi(A, B)$ in *loc. cit.* and its universal property induces an associative map of homotopy P_n-algebras

$$\underline{Hom}^\phi(A, B) \otimes \underline{Hom}^\psi(B, C) \longrightarrow \underline{Hom}^{\psi \circ \phi}(A, C)$$

which precisely gives the aforementionned n-bialgebra structure of $\underline{Hom}^{Id}(A, A)$.

The (homotopy) P_{n+1}-structure on $HH^\bullet_{\mathbb{P}_n}(A, A)$ is canonically induced by the n-bialgebra structure on $\underline{Hom}^{Id}(A, A)$, see [Ta2, Corollary 4.5 and §5], [CW] and the fact it corepresents the functor $F^{id}_{A,A}$.

Hence in order to prove the lemma we now need to prove that, for any map $f : A \to B$ between CDGAs, $CH^{S^n}(A, B)$ is isomorphic to $\underline{Hom}^\phi(A, B)$ as an homotopy P_n-algebra. Since both functors are functorial with respect to CDGA maps, we can further assume that A and B are free graded commutative as algebras. That is $A = (\mathrm{Sym}(V), d)$ and $B = (\mathrm{Sym}(W), b)$.

By definition, as a coalgebra $\mathbb{P}_n^\vee(A) = \mathrm{Sym}\big(\mathrm{CoLie}(A[-1])[1-n]\big)$ where CoLie is the free Lie coalgebra functor and Sym is endowed with the cofree coproartinian cocommutative cobracket. Note that $\mathrm{CoLie}(A[-1])$ is simply the underlying vector space of the Harrison chain complex of A (see [Ta, GiHa, L]). Since A is seen as a P_n-algebra with trivial bracket, the differential $\partial_{\mathbb{P}_n^\vee(A)}$ boils down to the usual Hochschild/Harisson complex differential. Hence, one has an isomorphism of complexes

$$\underline{Hom}^\phi(A, B) = Hom(\mathbb{P}_n^\vee(A), B)[-n] \cong Hom_A\Big(\mathrm{Sym}_A\big(\mathrm{Harr}_*(A, A)[1-n]\big), B\Big)[-n]$$

where the right hand side is endowed with the tensor product of Harrison differentials on A and consists of A-linear maps; here B is seen as an A-module through the map $\phi : A \to B$ and the A-module structure on $\mathrm{Harr}_*(A, A)$ is given by the

tensor product $A \otimes \mathrm{CoLie}(A[-1])$. The proof is the same as the one in the case $\phi = Id$ in [Ta, GiHa].

We recall that the Harrison chain complex $\mathrm{Harr}_*(A, A)$ is a sub-complex of the Hochschild chain complex of A: it is precisely the weight 1 part of the Hodge decomposition of the Hochschild complex of the differential graded commutative algebra A, see [L, GiHa]. Its homology is equal to the André-Quillen homology since we are in characteristic 0. Hence, since $A = (\mathrm{Sym}(V), d)$, by the Hochschild-Kostant-Rosenberg Theorem, the Harrison chain complex is quasi-isomorphic to $\Omega^1(A) = \mathrm{Sym}(V) \otimes \mathrm{Sym}(V[-1])$ where the differential is induced by the one on $S(V)$ and $d(v[-1]) = -s(d(v))$ where s is the unique derivation extending $v \mapsto v[-1]$ for $v \in V$ (cf. [L, § 5]).

It follows that we have a quasi-isomorphism of P_n-algebras

$$Hom_A\big(\mathrm{Sym}(V \oplus V[-n]), B\big) \overset{\simeq}{\longrightarrow} Hom(\mathbb{P}_n^\vee(A), B)[-n] = \underline{Hom}^\phi(A, B)$$

given by the convolution product on the left hand side $Hom_A\big(\mathrm{Sym}(V \oplus V[-n]), B\big) \cong Hom(\mathrm{Sym}(V[-n]), B)$ where $\mathrm{Sym}(V[-n])$ is seen as a P_n-coalgebra with trivial cobracket and its cofree cocommutative structure. Now the result follows from Lemma 4.17. $\qquad \square$

Combining the previous statements, we get:

Corollary 6.37. *Let A be a differential graded commutative algebra over a characteristic zero field. For $n \geq i \geq 2$, choose formality equivalences of ∞-operads $\mathcal{F}_i : \mathbb{E}_i \overset{\simeq}{\to} \mathbb{P}_i$. There is a canonical equivalence of E_{n+1}-algebras (and thus of P_{n+1}-algebras):*

$$(64) \qquad \mathrm{Sym}_A\big(\mathbb{R}\mathrm{Der}(A, A)[n]\big) \cong HH^\bullet_{\mathcal{E}_n}(A, A)$$

where the right hand side is induced by the E_{n+1}-algebra structure given by the Deligne conjecture (Theorem 6.28) and the left hand side is given the Schouten structure, with differential induced by the one in A.

Proof. By Theorem 6.35 we are left to prove that Tamarkin P_{n+1}-structure on $HH^\bullet_{\mathbb{P}_n}(A, A)$ is equivalent to the (homotopy) P_{n+1}-structure on $HH^\bullet_{\mathcal{E}_n}(A, A) \cong CH^{S^n}(A, A)$ provided by Theorem 6.31. This is precisely the content of Lemma 6.36. $\qquad \square$

Remark 6.38. Corollary 6.37 and Corollary 6.34 implies in particular that the P_n-cohomology $HH^\bullet_{\mathbb{P}_n}(A, A)$ of Tamarkin is equivalent as an $E_1 \odot P_n$-algebra to the $\mathbb{P}_n$-branes cohomology of Toën.

6.5.3. *Explicit computations using higher formality.* Now we assume $A = (S(V), d)$ is a *Sullivan algebra*, that is, as an algebra, it is the free graded commutative algebra on a graded vector space V and it is also equipped with a differential d. In that case the canonical map $\mathrm{Der}(A, A) \to \mathbb{R}\mathrm{Der}(A, A)$ is an equivalence of (dg-) Lie algebras. Hence, by Corollary 6.37 we have:

Corollary 6.39. *Let $A = (S(V), d)$ be a Sullivan algebra. Under the assumptions of Corollary 6.37, we have a canonical equivalence of E_{n+1}-algebras (and thus P_{n+1}-algebras as well):*

$$(65) \qquad \mathrm{Sym}_A\big(\mathrm{Der}(A, A)[n]\big) \cong HH^\bullet_{\mathcal{E}_n}(A, A).$$

Here the right hand side is has the E_{n+1}-algebra structure given by the Deligne conjecture (Theorem 6.28) and the left hand side is endowed with the structure corresponding to the Schouten algebra structure.

The main interest of Corollary 6.39 for us, is that the left hand side has a totally explicit and elementary strict P_{n+1}-algebra structure. It thus gives a *large class of examples of explicit computations* of the E_{n+1}-structure of centers of commutative algebras (viewed as E_n-algebras).

In the next three examples, we calculate the P_{n+1} structures of the left hand side of equation (65) for the cases of the Sullivan models of an odd sphere, an even sphere, and for the complex projective space.

Example 6.40 (P_{n+1}-structure (65) for the odd sphere S^{2k+1})**.** We compute the P_{n+1}-structure of (65) for the Sullivan algebra A of the $(2k+1)$-sphere. In this case, we consider the Sullivan algebra $A = (S(V), d)$ given by the free algebra generated by x in degree $|x| = 2k + 1$ with trivial differential $d = 0$. Since x is in odd degree, we have that $x^2 = 0$, so that $A = span\{1, x\}$ with the trivial algebra structure. Now note, that a (graded) derivation of A is uniquely determined by its value on the generator x, and that any derivation maps 1 to 0. We have essentially the two derivations of A, α and β, given by

$$\alpha(1) = 0, \quad \alpha(x) = 1, \qquad \text{and} \qquad \beta(1) = 0, \quad \beta(x) = x.$$

Note that $\beta = x.\alpha$, displaying $\mathrm{Der}(A, A)$ as a module over A. Furthermore, the Lie-bracket is calculated as the commutator,

$$(66) \qquad\qquad [\alpha, \beta] = \alpha, \quad \text{and} \quad [\alpha, \alpha] = [\beta, \beta] = 0.$$

This induces the bracket of $\mathrm{Der}(A, A)[n] = span\{\alpha, \beta\}$ after a shift by n, where α and β now have degrees $|\alpha| = n - (2k + 1)$ and $|\beta| = n$. Using this, we next calculate the P_{n+1}-structure on

$$\mathrm{Sym}_A(\mathrm{Der}(A, A)[n]) \;\; = \;\; A \oplus \mathrm{Der}(A, A)[n]$$
$$\oplus \big(\mathrm{Der}(A, A)[n] \odot_A \mathrm{Der}(A, A)[n]\big) \oplus \dots$$

Here we denote the algebra structure on $\mathrm{Sym}_A(\mathrm{Der}(A, A)[n])$ by "$\odot$" or "$\odot_A$" to indicate linearity over A. Since $\beta = x.\alpha$, note that any element of $\mathrm{Sym}_A(\mathrm{Der}(A, A)[n])$ is a sum of elements of the form $a.\alpha^{\odot p} = a.\alpha \odot_A \cdots \odot_A \alpha$ for some $a \in A$ and α is as above. The differential on $\mathrm{Sym}_A(\mathrm{Der}(A, A)[n])$ is zero, since $d = 0$. Recall the usual Poisson relation and anti-symmetry for the bracket in a P_{n+1}-algebra, *e.g.* from [SW, page 220],

$$(67) \qquad\qquad [f \odot g, h] \;\; = \;\; f \odot [g, h] + (-1)^{|g|(|h|+n)}[f, h] \odot g$$

$$(68) \qquad\qquad [f, g \odot h] \;\; = \;\; [f, g] \odot h + (-1)^{|g|(|f|+n)}g \odot [f, h],$$

$$(69) \qquad\qquad [f, g] \;\; = \;\; -(-1)^{(|f|+n)(|g|+n)}[g, f],$$

for $f, g, h \in \mathrm{Sym}_A(\mathrm{Der}(A, A)[n])$, as well as the Schouten identities,

$$(70) \qquad [\rho, a] \;\; = \;\; \rho(a) \qquad\quad \text{for } \rho \in \mathrm{Der}(A, A)[n], \text{ and } a \in A,$$

$$(71) \qquad [a, b] \;\; = \;\; 0 \qquad\qquad \text{for } a, b \in A,$$

which are used in defining the bracket on $\mathrm{Sym}_A(\mathrm{Der}(A, A)[n])$ together with (66). (Note that this gives indeed a well-defined bracket on $\mathrm{Sym}_A(\mathrm{Der}(A, A)[n])$ due to the commutator of derivations giving the consistency relation $[\rho, a.\lambda] = \rho \circ (a\lambda) -$

$(-1)^{|\rho|(|a|+|\lambda|)}(a\lambda)\circ\rho = \rho(a)\lambda + (-1)^{|\rho|\cdot|a|}a[\rho,\lambda]$ for $a \in A$ and $\rho,\lambda \in \mathrm{Der}(A,A)$ before shifting by n.)

In the case when n is odd, the degree $|\alpha| = n - (2k+1)$ is even, so that for any $a,b \in A$, and $p,q \in \mathbb{N}_0$,

$$
\begin{aligned}
[a.\alpha^{\odot p}, b.\alpha^{\odot q}] &= a[\alpha^{\odot p}, b\alpha^{\odot q}] + [a, b\alpha^{\odot q}]\odot\alpha^{\odot p} \\
&= a[\alpha^{\odot p}, b]\odot\alpha^{\odot q} + (-1)^{|b|}ab\underbrace{[\alpha^{\odot p}, \alpha^{\odot q}]}_{=0} \\
&\quad + \underbrace{[a,b]}_{=0}\alpha^{\odot p+q} + (-1)^{|b|(|a|+n)}b\underbrace{[a,\alpha^{\odot q}]}_{=-(-1)^{|a|+n}[\alpha^{\odot q},a]}\odot\alpha^{\odot p} \\
&= \left(p\cdot a\cdot\alpha(b) - (-1)^{(|b|+1)(|a|+1)}q\cdot\alpha(a)\cdot b\right)\alpha^{\odot(p+q-1)}
\end{aligned}
$$

Thus, we obtain the following brackets for $\alpha^{\odot p}$ and $x\alpha^{\odot p}$,

$$
\begin{aligned}
[\alpha^{\odot p}, \alpha^{\odot q}] &= 0, \\
[\alpha^{\odot p}, x\alpha^{\odot q}] &= -[x\alpha^{\odot q}, \alpha^{\odot p}] = p\cdot\alpha^{\odot(p+q-1)}, \\
[x\alpha^{\odot p}, x\alpha^{\odot q}] &= (p-q)\cdot x\alpha^{\odot(p+q-1)}.
\end{aligned}
$$

In the case where n is even, the degree of $|\alpha| = n - (2k+1)$ is odd, so that $\alpha\odot\alpha = 0$ in $\mathrm{Sym}_A(\mathrm{Der}(A,A)[n])$. Thus, $\mathrm{Sym}_A(\mathrm{Der}(A,A)[n]) = A \oplus \mathrm{Der}(A,A)[n]$ with n-bracket given by (71), (70), and (66),

$$[x,x] = [\alpha,\alpha] = [x\alpha, x\alpha] = 0, [\alpha, x] = 1, [\alpha, x\alpha] = \alpha, [x\alpha, x] = x\alpha(x) = x.$$

We note that the above example is consistent with the calculation of the sphere product, see Remark 7.20 below. We next consider the Sullivan algebra of the even sphere.

Example 6.41 (P_{n+1}-structure (65) for the even sphere S^{2k}). For $A = (S(V), d)$ the Sullivan algebra of the even $2k$-sphere, we calculate the left hand side of (65). More precisely, let A be the free algebra generated by x and y, where $|x| = 2k$ and $|y| = 4k - 1$. Since y is an odd element, it is $y^2 = 0$. The differential d is given by $d(x) = 0$ and $d(y) = x^2$. Any (graded) derivation of A is determined by its action on the generators x and y. For $\ell = 0, 1, 2, \ldots$, we can define derivations $\alpha_\ell, \beta_\ell, \gamma_\ell, \delta_\ell : A \to A$ of A whose actions on the generators are as follows.

$$
\begin{array}{lll}
(72) & \alpha_\ell(x) = x^\ell, & \alpha_\ell(y) = 0, \\
(73) & \beta_\ell(x) = x^\ell y, & \beta_\ell(y) = 0, \\
(74) & \gamma_\ell(x) = 0, & \gamma_\ell(y) = x^\ell, \\
(75) & \delta_\ell(x) = 0, & \delta_\ell(y) = x^\ell y.
\end{array}
$$

The degrees of these derivations $\alpha_\ell, \beta_\ell, \gamma_\ell, \delta_\ell \in \mathrm{Der}(A,A)[n]$ (after the shift by n) are

$$
\begin{aligned}
|\alpha_\ell| &= 2k(\ell - 1) + n, \\
|\beta_\ell| &= 2k(\ell - 1) + (4k - 1) + n = 2k(\ell + 1) + n - 1, \\
|\gamma_\ell| &= 2k\ell - (4k - 1) + n = 2k(\ell - 2) + n + 1, \\
|\delta_\ell| &= 2k\ell + n.
\end{aligned}
$$

Note, that any derivation can be written as a linear combination of the derivations $\alpha_\ell, \beta_\ell, \gamma_\ell$, and δ_ℓ. (In particular, $d = \gamma_2$.) Furthermore,

$$(76) \qquad x.\alpha_\ell = \alpha_{\ell+1}, \qquad y.\alpha_\ell = \beta_\ell, \qquad x.\gamma_\ell = \gamma_{\ell+1}, \qquad y.\gamma_\ell = \delta_\ell,$$

showing that $\mathrm{Der}(A, A)[n]$ is freely generated by α_0 and γ_0 as an A-module. The commutator in $\mathrm{Der}(A, A)[n]$ for α_0 and γ_0 is easily verified to vanish,

$$(77) \qquad\qquad\qquad [\alpha_0, \alpha_0] = [\alpha_0, \gamma_0] = [\gamma_0, \gamma_0] = 0.$$

(However, this does not imply that the bracket vanishes identically, since the bracket is not A-linear, but rather satisfies equations (67), (68), and (70). For example, it is $[\alpha_\ell, \alpha_m] = (m - \ell)\alpha_{\ell+m-1}$, etc.) The differential d of A induces a differential D on $\mathrm{Der}(A, A)[n]$ of degree $+1$ given by $D(\rho) = [d, \rho]$ for which we obtain $D(\alpha_\ell) = -2\gamma_{\ell+1}, D(\beta_\ell) = 2\delta_{\ell+1} + \alpha_{\ell+2}, D(\gamma_\ell) = 0, D(\delta_\ell) = \gamma_{\ell+2}$.

Since $\mathrm{Der}(A, A)[n]$ is generated by α_0 and γ_0 as an A-module, we see that any element of

$$\mathrm{Sym}_A\big(\mathrm{Der}(A, A)[n]\big) = A \oplus \mathrm{Der}(A, A)[n] \oplus \big(\mathrm{Der}(A, A)[n] \odot_A \mathrm{Der}(A, A)[n]\big) \oplus \ldots$$

can be written as a sum of terms of the form $a.\alpha_0^{\odot p} \odot \gamma_0^{\odot q}$ for $a \in A$ and $p, q \in \mathbb{N}_0$. We thus may obtain a differential D on $\mathrm{Sym}_A(\mathrm{Der}(A, A)[n])$ by taking

$$D(x) = 0, \quad D(y) = x^2, \quad D(\alpha_0) = -2\gamma_1 = -2x\gamma_0, \quad D(\gamma_0) = 0,$$

and extending this as a graded derivation (with respect to $\odot$).

Note that $[\alpha_0^{\odot p} \odot \gamma_0^{\odot q}, \alpha_0^{\odot r} \odot \gamma_0^{\odot s}] = 0$ by (67) and (68) and (77). In general, we have (with $|\alpha_0| \equiv n(\mathrm{mod}\ 2)$ and $|\gamma_0| \equiv n + 1(\mathrm{mod}\ 2)$):

$$(78) \quad [a.\alpha_0^{\odot p} \odot \gamma_0^{\odot q}, b.\alpha_0^{\odot r} \odot \gamma_0^{\odot s}]$$

$$= a[\alpha_0^{\odot p} \odot \gamma_0^{\odot q}, b.\alpha_0^{\odot r} \odot \gamma_0^{\odot s}] + (-1)^{\epsilon_1}[a, b.\alpha_0^{\odot r} \odot \gamma_0^{\odot s}] \odot \alpha_0^{\odot p} \odot \gamma_0^{\odot q}$$

$$= a[\alpha_0^{\odot p} \odot \gamma_0^{\odot q}, b] \odot \alpha_0^{\odot r} \odot \gamma_0^{\odot s} + (-1)^{\epsilon_1+\epsilon_2}b[a, \alpha_0^{\odot r} \odot \gamma_0^{\odot s}] \odot \alpha_0^{\odot p} \odot \gamma_0^{\odot q}$$

$$= a[\alpha_0^{\odot p} \odot \gamma_0^{\odot q}, b] \odot \alpha_0^{\odot r} \odot \gamma_0^{\odot s} - (-1)^{\epsilon_1+\epsilon_2+\epsilon_3}b[\alpha_0^{\odot r} \odot \gamma_0^{\odot s}, a] \odot \alpha_0^{\odot p} \odot \gamma_0^{\odot q},$$

where we used (67) in the first equality, (68) with $[\alpha_0^{\odot p} \odot \gamma_0^{\odot q}, \alpha_0^{\odot r} \odot \gamma_0^{\odot s}] = [a, b] = 0$ in the second equality, and (69) in the third equality. The signs are given as follows,

$$\begin{aligned}
\epsilon_1 &= (pn + q(n+1))(|b| + rn + s(n+1) + n), \\
\epsilon_2 &= |b|(|a| + n), \\
\epsilon_3 &= (|a| + n)(rn + s(n+1) + n), \\
\text{so that } \epsilon_1 + \epsilon_2 + \epsilon_3 &= (pn + q(n+1) + |a| + n)(rn + s(n+1) + |b| + n).
\end{aligned}$$

The right hand side of (78) may be evaluated further by evaluating $[\alpha_0^{\odot p} \odot \gamma_0^{\odot q}, b]$ and $[\alpha_0^{\odot r} \odot \gamma_0^{\odot s}, a]$ using equations (67) and (70).

To be more concrete, we now restrict to the case n being even. In this case α_0 is an even element while γ_0 is an odd element implying that $\gamma_0 \odot \gamma_0 = 0$, so that elements in $\mathrm{Sym}_A(\mathrm{Der}(A, A)[n])$ are either of the form $a.\alpha_0^{\odot p}$ or $a.\alpha_0^{\odot p} \odot \gamma_0$ for some $a \in A$. We obtain

$$[\alpha_0^{\odot p}, x^q] = pqx^{q-1}\alpha_0^{\odot(p-1)}, \qquad [\alpha_0^{\odot p} \odot \gamma_0, x^q] = pqx^{q-1}\alpha_0^{\odot(p-1)} \odot \gamma_0,$$

$$[\alpha_0^{\odot p}, x^q y] = pqx^{q-1}y\alpha_0^{\odot(p-1)}, \quad [\alpha_0^{\odot p} \odot \gamma_0, x^q y] = x^q\alpha_0^{\odot p} - pqx^{q-1}y\alpha_0^{\odot(p-1)} \odot \gamma_0.$$

This, together with (78) gives the following n-brackets:

(79)
$$\begin{cases} [x^r \alpha_0^{\odot p}, x^s \alpha_0^{\odot q}] & = (ps - qr) x^{r+s-1} \alpha_0^{\odot(p+q-1)}, \\ [x^r y \alpha_0^{\odot p}, x^s y \alpha_0^{\odot q}] & = 0, \\ [x^r \alpha_0^{\odot p} \odot \gamma_0, x^s \alpha_0^{\odot q} \odot \gamma_0] & = 0, \\ [x^r y \alpha_0^{\odot p} \odot \gamma_0, x^s y \alpha_0^{\odot q} \odot \gamma_0] & = 0, \end{cases}$$

and

(80)
$$\begin{cases} [x^r \alpha_0^{\odot p}, x^s y \alpha_0^{\odot q}] & = (ps - qr) x^{r+s-1} y \alpha_0^{\odot(p+q-1)}, \\ [x^r \alpha_0^{\odot p}, x^s \alpha_0^{\odot q} \odot \gamma_0] & = (ps - qr) x^{r+s-1} \alpha_0^{\odot(p+q-1)} \odot \gamma_0, \\ [x^r \alpha_0^{\odot p}, x^s y \alpha_0^{\odot q} \odot \gamma_0] & = (ps - qr) x^{r+s-1} y \alpha_0^{\odot(p+q-1)} \odot \gamma_0, \\ [x^r y \alpha_0^{\odot p}, x^s \alpha_0^{\odot q} \odot \gamma_0] & = (ps - qr) x^{r+s-1} y \alpha_0^{\odot(p+q-1)} \odot \gamma_0 + x^{r+s} \alpha_0^{\odot(p+q)}, \\ [x^r y \alpha_0^{\odot p}, x^s y \alpha_0^{\odot q} \odot \gamma_0] & = -x^{r+s} y \alpha_0^{\odot(p+q)}, \\ [x^r \alpha_0^{\odot p} \odot \gamma_0, x^s y \alpha_0^{\odot q} \odot \gamma_0] & = x^{r+s} \alpha_0^{\odot(p+q)} \odot \gamma_0. \end{cases}$$

Example 6.42 (P_{n+1}-structure (65) for the complex projective space $\mathbb{C}P^m$)**.** For the complex projective space $\mathbb{C}P^m$, the Sullivan model $A = (S(V), d)$ is generated by x in degree $|x| = 2$ and y in degree $|y| = 2m + 1$ with differential $d(x) = 0$ and $d(y) = x^{m+1}$. Note, that in this case we have again two generators x and y which have the same parity as in the generators x and y in the last Example 6.41 for the even sphere. Therefore, much of the arguments from Example 6.41 can be repeated with only minor modifications. First, note that $\mathrm{Der}(A, A)[n]$ is generated by the derivations $\alpha_\ell, \beta_\ell, \gamma_\ell, \delta_\ell$ given by formulas (72)-(75), however with x and y in the new degrees stated above. Thus, the degrees in the (shifted) space of derivations $\mathrm{Der}(A, A)[n]$ are now

$$|\alpha_\ell| = 2(\ell - 1) + n,$$
$$|\beta_\ell| = 2(\ell - 1) + (2m + 1) + n = 2(\ell + m) + n - 1,$$
$$|\gamma_\ell| = 2\ell - (2m + 1) + n = 2(\ell - m) + n - 1,$$
$$|\delta_\ell| = 2\ell + n.$$

The module relations (76) and the basic brackets (77) remain the same. However, the differential is now $d = \gamma_{m+1}$, so that D on $\mathrm{Der}(A, A)$ becomes $D(\rho) = [\gamma_{m+1}, \rho]$, which gives the relations $D(\alpha_\ell) = -(m+1)\gamma_{\ell+m}, D(\beta_\ell) = (m+1)\delta_{\ell+m} + \alpha_{\ell+m+1}, D(\gamma_\ell) = 0, D(\delta_\ell) = \gamma_{\ell+m+1}$. Thus D is defined on $\mathrm{Sym}_A(\mathrm{Der}(A, A)[n])$ by taking

$$D(x) = 0, \quad D(y) = x^{m+1}, \quad D(\alpha_0) = -(m+1)\gamma_m = -(m+1)x^m \gamma_0, \quad D(\gamma_0) = 0,$$

and extending this to $\mathrm{Sym}_A(\mathrm{Der}(A, A)[n])$ as a graded derivation. Also, the considerations concerning the n-bracket (such as equation (78) and, when n is even, equations (79) and (80)) apply just as in Example 6.41 for the even sphere.

7. Integral chain models for higher string topology operations

We will use the E_∞-Poincaré duality and Hochschild chains to give an algebraic model for Brane Topology at the *chain* level, over an arbitrary coefficient ring.

7.1. **Brane operations for n-connected Poincaré duality space.** Recall that
the n-dimensional free sphere space is denoted $X^{S^n} = Map(S^n, X)$. It is the
space of continuous map from S^n to X endowed with the compact-open topology.
Sullivan and Voronov [CV, Section 5] have shown that there is a natural graded
commutative algebra structure, called the *sphere product*, on the shifted homology
$H_{\bullet+\dim(M)}(M^{S^n})$ of an oriented closed manifold. For $n = 1$, this structure agrees
with the Chas-Sullivan loop product [CS]. This product was extended to all oriented
stacks in [BGNX]. For $n = 2$, the sphere product is a special case of the surface
product studied in [GTZ]. Further, it is claimed that $H_{\bullet+\dim(M)}(M^{S^n})$ is an algebra
over the homology $H_*(\mathcal{E}_{n+1}^{fr})$ of the framed little disk operad $\mathcal{E}_{n+1}^{fr}$. Below we
will forget about the $SO(n + 1)$-action and deal with action of the $\mathcal{E}_{n+1}$-operad
at the *chain* (and not homology) level and without specific assumptions on the
characteristic of the ground ring k.

We start by stating one of our main results:

Theorem 7.1. *Let X be an n-connected Poincaré duality space. Then the shifted
chain complex $C_{\bullet+\dim(X)}(X^{S^n})$ has a natural[41] E_{n+1}-algebra structure which in-
duces the sphere product* [CV, Section 5]

$$H_p\big(X^{S^n}\big) \otimes H_q\big(X^{S^n}\big) \to H_{p+q-\dim(X)}\big(X^{S^n}\big)$$

in homology when X is an oriented closed manifold.

Proof. Remark 5.25 implies that the homology groups of X are finitely generated
so that the biduality homomorphism $C_*(X) \to (C^*(X))^\vee$ is a quasi-isomorphism.
Since X is a Poincaré duality space, it then follows from Corollary 5.26 that the
Poincaré duality map (33)

$$\chi_X : C^*(X) \to C_*(X)[\dim(X)] \cong \big(C^*(X)\big)^\vee[\dim(X)]$$

is an equivalence of $C^*(X)$-E_∞-Modules. Thus it yields an equivalence

(81) $CH^{S^n}\left(C^*(X), C^*(X)\right) \cong Hom_{C^*(X)}\left(CH_{S^n}(C^*(X)), C^*(X)\right)$

$$\stackrel{(\chi_X)\circ-}{\longrightarrow} Hom_{C^*(X)}\left(CH_{S^n}(C^*(X)), \big(C^*(X)\big)^\vee\right)[\dim(X)]$$

$$\cong CH^{S^n}\left(C^*(X)), \big(C^*(X)\big)^\vee\right)[\dim(X)].$$

Since X is n-connected, by Corollary 3.36, there is an equivalence

(82) $CH^{S^n}\left(C^*(X)), \big(C^*(X)\big)^\vee\right) \cong C_*\big(X^{S^n}\big).$

Combining the equivalences (81) and (82), we get a natural equivalence

(83) $CH^{S^n}\left(C^*(X), C^*(X)\right) \cong C_*\big(X^{S^n}\big)[\dim(X)].$

By Theorem 6.28, $CH^{S^n}\left(C^*(X), C^*(X)\right)$ has a natural E_{n+1}-algebra structure,
whose underlying E_1-algebra structure is given by the cup-product. Hence the
equivalence (83) yields a natural E_{n+1}-structure on $C_*\big(X^{S^n}\big)[\dim(X)]$. Note that
the naturality with respect to maps $f : X \to Y$ of Poincaré duality spaces follows
from Theorem 7.10 below since a Poincaré duality space yields an object of $\mathcal{AM}$
and a map of Poincaré duality space is a map in $\mathcal{AM}$, see Example 7.8.(2) below.

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[41]with respect to maps of Poincaré duality spaces in the sense of Definition 5.27

From this observation follows the commutativity of the following diagram (in which $d = \dim(X) = \dim(Y)$)

$$
\begin{array}{ccc}
\left(CH^{S^n}\left(C^*(X), C^*(X)\right)\right)^{\otimes 2} & \xrightarrow{\quad \circ \quad} & CH^{S^n}\left(C^*(X), C^*(X)\right) \\
\downarrow{\scriptstyle (\chi_X)\circ -^{\otimes 2}} \; \cong & & \cong \; \downarrow{\scriptstyle (\chi_X)\circ -} \\
\left(CH^{S^n}\left(C^*(X), C_*(X)\right)[d]\right)^{\otimes 2} & & CH^{S^n}\left(C^*(X), C_*(X)\right)[d] \\
\downarrow{\scriptstyle (f_*)^{\otimes 2}} & & \downarrow{\scriptstyle f_*} \\
\left(CH^{S^n}\left(C^*(Y), C_*(Y)\right)[d]\right)^{\otimes 2} & & CH^{S^n}\left(C^*(Y), C_*(Y)\right)[d] \\
\uparrow{\scriptstyle (\chi_Y)\circ -^{\otimes 2}} \; \cong & & \cong \; \uparrow{\scriptstyle (\chi_Y)\circ -} \\
\left(CH^{S^n}\left(C^*(Y), C^*(Y)\right)\right)^{\otimes 2} & \xrightarrow{\quad \circ \quad} & CH^{S^n}\left(C^*(Y), C^*(Y)\right)
\end{array}
$$

where the horizontal arrows are given by the composition (59) of (derived) homomorphisms (and Proposition 6.2). By Theorem 7.10, the vertical maps are maps of E_n-algebras. Thus the above diagram shows that a map of Poincaré duality space induces a map of E_1-algebras (with respect to the composition (59)) in the (symmetric monoidal) category of E_n-algebras and thus induces a map of E_{n+1}-algebras by Dunn Theorem (see [Du, L-HA] or Theorem 2.30): $E_1 - Alg\big(E_n - Alg\big) \cong E_{n+1} - Alg$.

It remains to identify the underlying multiplication in homology with its purely topological counterpart. This is done in Section 7.2, see Proposition 7.17. $\square$

Passing to homology in Theorem 7.1, we recover the following result first stated in [CV].

Corollary 7.2. *Let X be a n-connected Poincaré duality space. Then the shifted homology $H_{\bullet + \dim(X)}(X^{S^n})$ has a natural P_{n+1}-algebra[42] structure which induces the sphere product* [CV, Section 5]

$$
H_p\big(X^{S^n}\big) \otimes H_q\big(X^{S^n}\big) \to H_{p+q-\dim(X)}\big(X^{S^n}\big)
$$

in homology when X is an oriented closed manifold.

Remark 7.3. Theorem 7.1 (as well as Corollary 7.6 below) still holds if X is a Poincaré duality space which is connected, nilpotent with finite homotopy groups in degree less than or equal to n. This is seen by using Proposition 3.38 in addition to Corollary 3.36 in the proof of the Theorem.

Example 7.4 (Explicit computation in characteristic zero)**.** Let $n \geq 2$. In characteristic zero, the singular cochains on X are equivalent, as an E_∞-algebra, to a Sullivan algebra (as in § 6.5.3) and, in particular, one can compute the Brane topology structure given by Theorem 7.1 using Corollary 6.39 which gives very explicit combinatorial models.

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[42]such that is the induced Lie algebra structure is the one of a restricted Lie algebra

Example 7.5. Assume M is a simply connected closed manifold. Then Theorem 7.1 yields an E_2-structure on the chains $C_*(LM)[\dim(M)]$ of the free loop space LM, thus string topology operations at the chain level. According to Example 6.29 and Proposition 7.17 below, the underlying Gerstenhaber structure is the classical Chas-Sullivan one [CS].

Corollary 7.6. *Let X, Y be n-connected ($n \geq 1$) closed manifolds of the same dimension and assume $f : M \to N$ induces an isomorphism in homology such that $f_*([X]) = [Y] \in H_*(Y, k)$. Then the induced bijection $H_*(X^{S^n}) \cong H_*(Y^{S^n})$ is an algebra isomorphism (with respect to the sphere product).*

In particular, the sphere product is an homotopy invariant of n-connected manifolds (with respect to orientation preserving maps).

Proof. By assumption, the induced map $\cap f_*([X]) : C^*(Y) \to C_*(Y)[\dim(Y)]$ and $\cap [Y] C^*(Y) \to C_*(Y)[\dim(Y)]$ are homotopic. Thus f induces a map of Poincaré duality spaces $(X, [X]) \to (Y, [Y]))$ which is a quasi-isomorphism. Then, by Theorem 7.1, $f_* : C_{*+\dim(X)}(X^{S^n}) \to C_{*+\dim(Y)}(Y^{S^n})$ is an equivalence of E_n-algebras. In particular, it is an algebra isomorphism in homology so that the result follows from the identification of the sphere product as asserted in Theorem 7.1 (see Proposition 7.17). $\square$

The above brane product fits into a larger setting of setups[43] to define $\mathcal{E}_n$-actions on $CH^{S^n}(A, M)$. In fact, we start with the following general setup.

Definition 7.7. We define $\mathcal{AM}$ as the following category. The objects of $\mathcal{AM}$ are triples (A, M, μ), where A is an E_∞-algebra, M is an E_∞-A-module, and, considering the E_∞-algebra $A \otimes A$ with canonical E_∞-$(A \otimes A)$-modules M and $M \otimes M$ (induced via the E_∞ structure map $A \otimes A \to A$), we assume that $\mu : M \otimes M \to M$ is an E_∞-$(A \otimes A)$-module map[44]. The morphisms of $\mathcal{AM}$ consist of tuples $(f, g) : (A, M, \mu) \to (A', M', \mu')$, where $f : A \to A'$ is an E_∞-morphism, thus inducing an E_∞-A-module structure on M', and $g : M' \to M$ is an E_∞-A-module map, satisfying the compatibility relation,

$$(84) \qquad \begin{array}{ccc} M' \otimes M' & \xrightarrow{\mu'} & M' \\ {\scriptstyle g \otimes g} \downarrow & & \downarrow {\scriptstyle g} \\ M \otimes M & \xrightarrow{\mu} & M \end{array}$$

in k-Mod_∞.

There are two main examples we have in mind for the above definition.

Example 7.8.   (1) The first example relates to the sphere product as considered in Section 4.2 and also in [G1]. Let A and B be two E_∞-algebras, and let $h : A \to B$ be a morphism of E_∞-algebras. Then, h makes $M := B$ into an E_∞-A-module, and the E_∞ structure of B gives a map $B \otimes B \to B$ which is also an E_∞-$(A \otimes A)$-module map. Furthermore, if h factors through an E_∞-algebra B' as a composition of E_∞-algebras

---

[43]which is useful to study functoriality of brane operations

[44]said otherwise, the objects of $\mathcal{AM}$ are the objects N of the monoidal ∞-category Mod^{E_n} endowed with a structure map $\mu_N : N \otimes N \to N$; the morphisms are however different

maps $h : A \xrightarrow{h'} B' \xrightarrow{g} B$, then this induces a morphism between the spaces $(id_A, g) : (A, B, \mu) \to (A, B', \mu')$.

(2) The second example relates to generalizations of sphere topology products as described in Theorem 7.1 above. Let A be an E_∞-algebra and M be an E_∞-A-module and given an E_∞-module map $\rho : M \to A$. We define the induced E_∞-$(A \otimes A)$-module map $\mu : M \otimes M \to M$ as the composition of ρ and the E_∞-A-module structure of M,

$$\mu : M \otimes M \xrightarrow{\rho \otimes id} A \otimes M \longrightarrow M.$$

Furthermore, any map of two given E_∞-A-modules $g : M' \to M$ which commutes with E_∞-A-module maps ρ and ρ',

also respects the induced relation (84), since $g \circ \mu'(m'_1, m'_2) = g(\rho'(m'_1).m'_2) = \rho'(m'_1).g(m'_2) = \rho(g((m'_1)).g(m'_2) = \mu \circ (g \otimes g)(m'_1, m'_2)$.

For example, consider the setup from Section 5.4: $C_*(X)$ is an E_∞-coalgebra, $C^*(X) = Hom_k(C_*(X), k)$ is its linear dual endowed with its canonical E_∞-algebra structure, and caping with the fundamental cycle $\cap [X] : C^*(X) \to C_*(X)[\dim(X)]$ induces an E_∞-quasi-isomorphism of E_∞-A-modules. The quasi-inverse of this map is an E_∞-A-module map $\rho : M := C_*(X)[\dim(X)] \to A := C^*(X)$. Moreover, if $f : (X, [X]) \to (Y, [Y])$ is a map of Poincaré duality space (Definition 5.27), then the tuple $(f^* : C^*(Y) \to C^*(X), f_* : C_*(X) \to C_*(Y))$ is a map in the category $\mathcal{AM}$.

For any triple (A, M, μ) which is an object of $\mathcal{AM}$ described in Definition 7.7, we can consider the Hochschild cochains $CH^{S^d}(A, M)$. We claim that there is an E_d-algebra structure on $CH^{S^d}(A, M)$, generalizing the E_d-algebra structure from Theorem 4.12.

Definition 7.9. Using the notation from Section 4.2, we define the E_d-algebra structure on $CH^{S^d}(A, M)$ by,

$$C_*\big(\mathcal{C}_d(r)\big) \otimes \Big(CH^{S^d}(A, M)\Big)^{\otimes r} \longrightarrow C_*\big(\mathcal{C}_d(r)\big) \otimes \Big(Hom_A(A^{\otimes S^d}, M)\Big)^{\otimes r}$$

$$\longrightarrow C_*\big(\mathcal{C}_d(r)\big) \otimes Hom_{A^{\otimes r}}((A^{\otimes S^d})^{\otimes r}, M^{\otimes r})$$

$$\stackrel{id\otimes(\mu^{\circ(r-1)})_*}{\longrightarrow} C_*\big(\mathcal{C}_d(r)\big) \otimes Hom_{A^{\otimes r}}((A^{\otimes S^d})^{\otimes r}, M)$$

$$\stackrel{\cong}{\longrightarrow} C_*\big(\mathcal{C}_d(r)\big) \otimes Hom_{A^{\otimes r}}((A^{\otimes S^d})^{\otimes r}, Hom_A(A, M))$$

$$\stackrel{\cong}{\longrightarrow} C_*\big(\mathcal{C}_d(r)\big) \otimes Hom_A(A \otimes^{\mathbb{L}}_{A^{\otimes r}} (A^{\otimes S^d})^{\otimes r}, M)$$

$$\stackrel{\cong}{\longrightarrow} C_*\big(\mathcal{C}_d(r)\big) \otimes Hom_A(A^{\otimes(\overbrace{S^d \vee \cdots \vee S^d}^{r \text{ times}})}, M)$$

$$\longrightarrow C_*\big(\mathcal{C}_d(r)\big) \otimes CH^{S^d \vee \cdots \vee S^d}(A, M) \stackrel{pinch^*}{\longrightarrow} CH^{S^d}(A, M).$$

We need to show compatibility of the involved operad action. This is similar to the proof in section 4.2.

In fact, more is true:

Theorem 7.10. *The identification given in the previous Definition 7.9 defines a (contravariant) functor $CH^{S^d} : \mathcal{AM} \to E_d - Alg$.*

Proof. It only remains to show that morphisms $(f, g) : (A, M, \mu) \to (A', M', \mu')$ in $\mathcal{AM}$ induce maps of E_d-algebras. Since $f : A \to A'$ makes M' into an E_∞-A-algebra, and with this $\mu' : M' \otimes M' \to M'$ into a map of E_∞-$(A \otimes A)$-modules, this follows from the commutativity of the following diagram:

By the virtue of the previous theorem and Example 7.8(2), we can thus define a family of sphere topology operations, one for each E_∞-module map $C_*(X)[\dim(X)] \to C^*(X)$, which are related by morphisms of E_d-algebras.

In particular, for $d = 1$, we can obtain (chain level, characteristic free) *string topology* operations associated to any E_∞-module map $C_*(M)[\dim(M)] \to C^*(M)$.

7.2. Topological identification of the brane product.

In this section, we prove that the cup product of Hochschild cochains over spheres identifies with the usual "brane product" in the homology of a free sphere space. The idea of the proof follows the surface product kind of proof from [GTZ, Theorem 3.4.2].

We start by recalling the construction of the sphere product of Sullivan-Voronov [CV]. Let M be a manifold equipped with a Riemannian metric and let the sphere spaces $Map(S^n, M)$ be equipped with Fréchet manifold structures. We further assume that M is closed, oriented. We have a cartesian square of fibrations

$$
(85) \qquad \begin{array}{ccc}
Map(S^n \vee S^n, M) & \xrightarrow{\rho_{in}} & Map(S^n, M) \times Map(S^n, M) \\
\downarrow & & \downarrow{\scriptstyle ev \times ev} \\
M & \xrightarrow{\text{diagonal}} & M \times M
\end{array}
$$

where the evaluation maps on the right are furthermore submersions. We denote $Tub(M) \subset M \times M$ a tubular neighborhood of the diagonal of M, which can be identified to the normal bundle of the diagonal. The pullback $(ev \times ev)^{-1}(Tub(M))$ by the submersion $ev \times ev : Map(S^n, M) \times Map(S^n, M) \to M \times M$ can be identified with a tubular neighborhood $Tub(Map(S^n \vee S^n, M))$ of ρ_{in} and thus with a normal bundle of ρ_{in}. One forms the corresponding Thom spaces M^{-TM} and $Map(S^n \vee S^n, M)^{-TM}$ by collapsing all the complements of the tubular neighborhood to a point. These Thom spaces are spheres (of dimension $\dim(M)$) bundles over, respectively M, and $Map(S^n \vee S^n, M)$. Hence, we have a diagram of pullback squares

$$
\begin{array}{ccc}
Map(S^n \coprod S^n, M) \xrightarrow{\text{collapse}} Map(S^n \vee S^n, M)^{-TM} & \xrightarrow{\pi} & Map(S^n \vee S^n, M) \\
\downarrow{\scriptstyle ev \times ev} \qquad\qquad\qquad \downarrow{\scriptstyle ev} & & \downarrow{\scriptstyle ev} \\
M \times M \xrightarrow{\quad\text{collapse}\quad} M^{-TM} & \xrightarrow{\quad\pi\quad} & M
\end{array}
$$

where the vertical arrows are fibrations. In particular, the Thom class of ρ_{in} is the pullback $(ev^*)(th(M)) \in H^{\dim(M)}(Map(S^n \vee S^n, M)^{-TM})$ of the Thom class $th(M) \in H^{\dim(M)}(M^{-TM})$ of $M \to M \times M$.

The above setup allows us to define a Gysin map

$$
(\rho_{in})_! : H_*\left(M^{S^n \coprod S^n}\right) \longrightarrow H_{*-\dim(M)}\left(M^{S^n \vee S^n}\right)
$$

as the composition

$$
(86) \qquad\qquad (\rho_{in})_! = \pi_* \circ (- \cap ev^*(th(M))) \circ (\text{collapse})_*.
$$

Definition 7.11 (Sullivan-Voronov [CV]). The sphere product is the composition

$$
\star_{S^n} : H_{*+\dim(M)}\left(M^{S^n}\right)^{\otimes 2} \to H_{*+2\dim(M)}\left(M^{S^n \coprod S^n}\right)
$$

$$
\xrightarrow{(\rho_{in})_!} H_{*+\dim(M)}\left(M^{S^n \vee S^n}\right) \xrightarrow{(\delta_{S^n}^*)_*} H_{*+\dim(M)}\left(M^{S^n}\right)
$$

where $\delta_{S^n} : S^n \to S^n \vee S^n$ is the pinching map.

Note that the Thom class $th(M)$ can be represented by any cocycle $t(M)$ which is Poincaré dual to the pushforward of the fundamental cycle $[M]$ of M, *i.e.*, $\chi_{M \times M}(\text{collapse}_*(t(M)) = (\text{diagonal}_*([M]))$ or, equivalently,

$$\chi_{M^{-TM}}(t(M)) = (\text{collapse} \circ \text{diagonal}_*([M])).$$

By Corollary 5.22, we get maps of E_∞-modules

$$\rho_{th(M)} : C_*(M^{-TM}) \longrightarrow C_{*-\dim(M)}(M^{-TM}),$$

$$\rho_{ev^*(th(M))} : C_*\big(Map(S^n \vee S^n, M)^{-TM}\big) \longrightarrow C_{*-\dim(M)}\big(Map(S^n \vee S^n, M)^{-TM}\big)$$

lifting the cap-products $-\cap t(M)$ and $-\cap ev^*(t(M))$. Thus we obtain the following chain level interpretation of the sphere product.

Lemma 7.12. *The sphere product (Definition 7.11) is induced by passing to the homology groups in the following composition*

$$(87) \quad \star_{S^n} : \big(C_*\big(M^{S^n}\big)[\dim(M)]\big)^{\otimes 2} \to C_*\big(M^{S^n \amalg S^n}\big)[2\dim(M)]$$

$$\xrightarrow{collapse_*} C_*\big((M^{S^n \vee S^n})^{-TM}\big)[2\dim(M)] \xrightarrow{\rho_{ev^*(th(M))}} C_*\big((M^{S^n \vee S^n})^{-TM}\big)[\dim(M)]$$

$$\xrightarrow{\pi_*} C_*\big(M^{S^n \vee S^n}\big)[\dim(M)] \xrightarrow{(\delta_{S^n}^*)_*} C_*\big(M^{S^n}\big)[\dim(M)].$$

Remark 7.13. In this section we only identify the sphere product which is the degree 0-component of a higher framed E_{n+1}-structure claimed in [CV, Section 5]. The reason is that we do not know higher degree representative of this operations (in a way similar to the map (87)) since such higher operations would involve a careful analysis of Gysin maps associated to higher cacti *in families*. However, it is possible that the new operads introduced by Bargheer in [Ba] could lead in a near future to explicit representatives of the degree n Lie Bracket in homology.

We now further assume X is a general Poincaré duality space (see Definition 5.23).

Recall that by Corollary 3.36 and Corollary 5.22, we have the equivalence (83):

$$CH^{S^n}(C^*(X), C^*(X)) \cong C_*\big(X^{S^n}\big)[\dim(X)].$$

The cup-product can be thus transfered (through the above equivalence) to give a multiplication $\big(C_*\big(X^{S^n}\big)[\dim(X)]\big)^{\otimes 2} \to C_*\big(X^{S^n}\big)[\dim(X)]$. We first wish to give another chain level representative for this multiplication, which is essentially the content of Lemma 7.15 below. We will then compare it with the sphere product $\star_{S^n}$ given by the composition (87).

The E_∞-algebra map $C^*(X \times X) \xrightarrow{diag^*} C^*(X)$ induced by the diagonal $X \to X \times X$ makes $C_*(X)$ an E_∞-$C^*(X \times X)$-module. By functoriality of the cap-product, the diagonal $C_*(X) \to C_*(X \times X)$ is a map of left $(C^*(X \times X), \cup)$-module.

By Theorem 5.13, we thus get a unique lift $C_*(X) \xrightarrow{diag_*} C_*(X \times X)$ of the diagonal map in $C_*(X \times X)\text{-}Mod^{E_\infty}$. By Lemma 3.30, there is an equivalence of E_∞-algebras $C^*(X \times X) \cong C^*(X) \otimes C^*(X)$. Further, Poincaré duality (Corollary 5.26) gives equivalences of E_∞-$C^*(Y)$-modules $\chi_X : C^*(Y) \xrightarrow{\simeq} C_*(Y)[\dim(Y)]$ for any Poincaré duality space Y.

Putting together the last three statements we obtain the first assertion in

Lemma 7.14. *Let X be a Poincaré duality space. There is a map in $C^*(X) \otimes C^*(X) - Mod^{E_\infty}$ given by the following composition:*

$$\nabla_X : C^*(X) \overset{\cong}{\to} C_*(X)[\dim(X)] \overset{diag_*}{\longrightarrow} C_*(X \times X)[\dim(X)]$$
$$\overset{\cong}{\to} C_*(X \times X)[\dim(X)] \overset{\cong}{\leftarrow} C^*(X \times X)[-\dim(X)]$$
$$\cong C^*(X) \otimes C^*(X)[-\dim(X)].$$

Further, for any closed oriented manifold M, the following diagram is commutative

$$C^*(M) \overset{\pi^*}{\longrightarrow} C^*(M^{-TM}) \overset{\rho^\vee_{th(M)}}{\longrightarrow} C^*(M^{-TM})[-\dim(M)] \overset{collapse^*}{\longrightarrow} C^*(M \times M)[-\dim(M)]$$

$$\nabla_M \qquad\qquad \downarrow{\cong}$$

$$C^*(M) \otimes C^*(M)[-\dim(M)]$$

in $C^*(X) \otimes C^*(X) - Mod^{E_\infty}$.

Proof. The second assertion follows from the identity

$$\mathrm{collapse}^*\big(\pi^*(x)\big) \cap \big(\mathrm{collapse}^*\big(t(M)\big) \cap [M \times M]\big) \;=\; \mathrm{collapse}^*\big(\pi^*(x)\big) \cap \mathrm{diagonal}_*([M])$$
$$=\; \mathrm{diagonal}_*(x \cap [M])$$

which follows from $\pi \circ \mathrm{collapse} \circ \mathrm{diagonal} = id$ and the definition of the Thom class. $\qquad\square$

It follows that the map ∇_X yields a map of $C^*(X)$-E_∞-modules

$$(88) \quad \nabla_{X*} : CH_{S^n \vee S^n}(C^*(X)) \cong CH_{S^n \coprod S^n}(C^*(X)) \overset{\mathbb{L}}{\underset{C^*(X)^{\otimes 2}}{\otimes}} C^*(X)$$

$$\overset{1 \otimes \nabla_X}{\longrightarrow} CH_{S^n \coprod S^n}(C^*(X)) \overset{\mathbb{L}}{\underset{C^*(X)^{\otimes 2}}{\otimes}} C^*(X)^{\otimes 2}[-\dim(X)]$$

$$\cong CH_{S^n \coprod S^n}(C^*(X))[-\dim(X)].$$

Thus, dualizing, we get a map

$$(89)$$
$$\nabla^! : CH^{S^n \coprod S^n}\Big(C^*(X), (C^*(X))^\vee\Big) \cong Hom_{C^*(X)}\Big(CH_{S^n \coprod S^n}(C^*(X)), (C^*(X))^\vee\Big)$$

$$\overset{\nabla_X^*}{\longrightarrow} Hom_{C^*(X)}\Big(CH_{S^n \vee S^n}(C^*(X)), (C^*(X))^\vee\Big)[-\dim(X)]$$

$$\cong CH^{S^n \vee S^n}\Big(C^*(X), (C^*(X))^\vee\Big)[-\dim(X)].$$

Recall that we have a pinching map $\delta_{S^n} : S^n \to S^n \vee S^n$ induced by collapsing the equator of S^n to a point. This gives us a multiplication

(90)

$$\mu_{S^n} : CH^{S^n}\left(C^*(X), (C^*(X))^\vee\right)^{\otimes 2} \cong Hom_{C^*(X)}\left(CH_{S^n}(C^*(X)), (C^*(X))^\vee\right)^{\otimes 2}$$

$$\longrightarrow Hom_{C^*(X)^{\otimes 2}}\left(CH_{S^n \coprod S^n}(C^*(X)), (C^*(X))^\vee \otimes (C^*(X))^\vee\right)$$

$$\cong Hom_{C^*(X)}\left(CH_{S^n \coprod S^n}(C^*(X)), (C^*(X))^\vee\right)$$

$$\xrightarrow{\nabla^!} CH^{S^n \vee S^n}\left(C^*(X), (C^*(X))^\vee\right)[-\dim(X)]$$

$$\xrightarrow{\delta^*_{S^n}} CH^{S^n}\left(C^*(X), (C^*(X))^\vee\right)[-\dim(X)].$$

Lemma 7.15. *Let X be a Poincaré duality space. There is a commutative (in $k\text{-Mod}_\infty$) diagram*

$$
\begin{array}{ccc}
CH^{S^n}\left(C^*(X), C^*(X)\right)^{\otimes 2} & \xrightarrow{\cup_{S^n}} & CH^{S^n}\left(C^*(X), C^*(X)\right) \\
\cong \downarrow & & \downarrow \cong \\
\left(CH^{S^n}\left(C^*(X), (C^*(X))^\vee\right)[\dim(X)]\right)^{\otimes 2} & \xrightarrow{\mu_{S^n}} & CH^{S^n}\left(C^*(X), (C^*(X))^\vee\right)[\dim(X)]
\end{array}
$$

where the top arrow is the sphere cup-product of Corollary 4.6 and the vertical arrows are induced by the Poincaré duality map $\chi_X : C^*(X) \to C_*(X)[\dim(X)] \to \left(C^*(X)\right)^\vee[\dim(X)]$.

Proof. By Lemma 3.30, the E_∞-algebra map $m_X : C^*(X) \otimes C^*(X) \to C^*(X)$ is the composition

$$m_X : C^*(X) \otimes C^*(X) \xrightarrow{AW^\vee} C^*(X \times X) \xrightarrow{diag^*} C^*(X).$$

It follows that the map ∇_X defined in Lemma 7.14 sits inside a commutative diagram

$$
\begin{array}{ccc}
C^*(X) \otimes C^*(X) & \xrightarrow{m_X} & C^*(X) \\
\chi_X \otimes \chi_X \downarrow & & \downarrow \chi_X \\
\left(C^*(X)\right)^\vee \otimes \left(C^*(X)\right)^\vee [2\dim(X)] & \xrightarrow{(\nabla_X)^\vee} & \left(C^*(X)\right)^\vee [\dim(X)]
\end{array}
$$

in $C^*(X) \otimes C^*(X) - Mod^{E_\infty}$. It follows that we get a commutative diagram
(91)

$$
\begin{array}{ccc}
Hom_{C^*(X)^{\otimes 2}}\left(CH_{S^n \coprod S^n}(C^*(X)), C^*(X)^{\otimes 2}\right) & \xrightarrow{(m_X)_*} & Hom_{C^*(X)^{\otimes 2}}\left(CH_{S^n \coprod S^n}(C^*(X)), C^*(X)\right) \\
(\chi_X)_*^{\otimes 2} \downarrow & & \downarrow (\chi_X)_* \\
Hom_{C^*(X)^{\otimes 2}}\left(CH_{S^n \coprod S^n}(C^*(X)), (C^*(X))^\vee)^{\otimes 2}\right) & \xrightarrow{(\nabla_X)_*^\vee} & Hom_{C^*(X)^{\otimes 2}}\left(CH_{S^n \coprod S^n}(C^*(X)), (C^*(X))^\vee\right) \\
\cong \downarrow & & \downarrow \cong \\
Hom_{C^*(X)}\left(CH_{S^n \coprod S^n}(C^*(X)), (C^*(X))^\vee\right) & \xrightarrow{\nabla^!} & Hom_{C^*(X)}\left(CH_{S^n \vee S^n}(C^*(X)), (C^*(X))^\vee\right)
\end{array}
$$

in k-Mod$_\infty$ (note that we have suppress the degree shifting in the diagram for simplicity). By functoriality, we have a commutative diagram

$$
\begin{array}{ccc}
Hom_{C^*(X)}\Big(CH_{S^n \vee S^n}(C^*(X)), C^*(X)\Big) & \xrightarrow{\delta^*_{S^n}} & Hom_{C^*(X)}\Big(CH_{S^n}(C^*(X)), C^*(X)\Big) \\
\Big\downarrow (\chi_X)_* & & \Big\downarrow (\chi_X)_* \\
Hom_{C^*(X)}\Big(CH_{S^n \vee S^n}(C^*(X)), (C^*(X))^\vee\Big) & \xrightarrow{\delta^*_{S^n}} & Hom_{C^*(X)}\Big(CH_{S^n}(C^*(X)), (C^*(X))^\vee\Big)
\end{array}
$$

which, together with the previous diagram (91) and the definition of the map $\cup_{S^n}$ (see Corollary 4.6) and the map (90), implies the Lemma. $\square$

The cartesian square of fibrations (85) shows that, when M is n-connected (and thus M^{S^n} is path connected), there is a quasi-isomorphism

(92)
$$
C^*\big(M^{S^n \vee S^n}\big) \cong C^*\big(M^{S^n} \amalg S^n\big) \overset{\mathbb{L}}{\underset{C^*\big(M \times M\big)}{\otimes}} C^*(M)
$$

so that the map ∇_M of Lemma 7.14 yields a map

$$
id \otimes \nabla_M : C^*\big(M^{S^n \vee S^n}\big) \cong C^*\big(M \times M\big) \overset{\mathbb{L}}{\underset{C^*\big(M \times M\big)}{\otimes}} C^*(M)
$$

$$
\overset{id \otimes_{C^*\big(M \times M\big)} \nabla_M}{\longrightarrow} C^*\big(M \times M\big)[-\dim(M)] \cong C^*\big(M^{S^n}\big)^{\otimes 2}[-\dim(M)].
$$

Lemma 7.16. *Let X be a n-connected Poincaré duality space. The following diagram*

$$
\begin{array}{ccc}
CH_{S^n}\big(C^*(X)\big) \xrightarrow{\left(\delta_{S^n}\right)_*} CH_{S^n \vee S^n}\big(C^*(X)\big) \xrightarrow{\nabla_{X*}} CH_{S^n}\big(C^*(X)\big)^{\otimes 2}[-\dim(X)] \\
\Big\downarrow \mathcal{I}t \qquad\qquad\qquad\quad \Big\downarrow \mathcal{I}t \qquad\qquad\qquad\qquad \Big\downarrow \mathcal{I}t^{\otimes 2} \\
C^*\big(X^{S^n}\big) \xrightarrow{\delta^*_{S^n}} C^*\big(X^{S^n \vee S^n}\big) \xrightarrow{id \otimes \nabla_X} C^*\big(X^{S^n}\big)^{\otimes 2}[-\dim(X)]
\end{array}
$$

is commutative in k-Mod$_\infty$ (here the map ∇_{X*} is the map (88)).

Proof. This is a consequence of the naturality of the map $\mathcal{I}t : CH_X(C^*(Y)) \to C^*(Y^X)$, see Corollary 3.35. $\square$

Proposition 7.17. *Let M be an n-connected oriented closed manifold. Then the following diagram*

$$
\begin{array}{ccc}
CH^{S^n}\big(C^*(X), C^*(X)\big)^{\otimes 2} & \xrightarrow{\cup_{S^n}} & CH^{S^n}\big(C^*(X), C^*(X)\big) \\
\cong \Big\downarrow & & \Big\downarrow \cong \\
\big(C_*\big(X^{S^n}\big)[\dim(X)]\big)^{\otimes 2} & \xrightarrow{\star_{S^n}} & C_*\big(X^{S^n}\big)[\dim(X)]
\end{array}
$$

is commutative in k-Mod$_\infty$. Here the horizontal arrows are the sphere cup-product of Corollary 4.6 and the sphere product (87); the vertical arrows are given by the equivalences (83) (induced by the Poincaré duality map and Corollary 3.36).

Since the vertical arrows are the maps defining the E_{n+1}-structure given by Theorem 7.1 on $C^*(X^{S^n})[\dim(X)]$; it follows that the underlying commutative algebra structure on homology agrees with the sphere product.

Proof. Recall that there is a canonical isomorphism $CH^{S^n}(A, A^\vee) \cong \left(CH_{S^n}(A, A^\vee)\right)^\vee$. By assumption M is a Poincaré duality space, hence it has finitely generated homology groups (Remark 5.25) and the canonical biduality map $C_*(X) \to (C^*(X))^\vee$ is a quasi-isomorphism and it is sufficient to prove that the dual of the diagram depicted in Proposition 7.17 is commutative.

By definition (90) of μ_{S^n}, lemma 7.16 and lemma 7.15 we are left to prove that the map $id \otimes \nabla_M : C^*(M^{S^n \vee S^n}) \to C^*(M^{S^n})^{\otimes 2}[-\dim(M)]$ sits inside a commutative diagram

(93)

$$C^*(M^{S^n \vee S^n}) \xrightarrow{\pi^*} C^*((M^{S^n \vee S^n})^{-TM}) \xrightarrow{\rho^\vee_{ev^*(th(M))}} C^*((M^{S^n \vee S^n})^{-TM})[-\dim(M)]$$

$$\downarrow{\text{collapse}^*}$$

$$C^*(M^{S^n \amalg S^n})[-\dim(M)]$$

$$\downarrow{\cong}$$

$$\xrightarrow{id \otimes \nabla_M} \qquad C^*(M^{S^n})^{\otimes 2}[-\dim(M)]$$

in $k\text{-Mod}_\infty$.

Recall the equivalence (92) above. Under this equivalence, the cup-product by the pullback $ev^*(t(M))$ is given by

$$C^*(M^{S^n \vee S^n}) \cong C^*(M^{S^n \amalg S^n}) \overset{\mathbb{L}}{\underset{C^*(M \times M)}{\otimes}} C^*(M)$$

$$\xrightarrow{id \otimes^{\mathbb{L}}_{C^*(M \times M)} \cup t(M)} C^*(M^{S^n \amalg S^n}) \overset{\mathbb{L}}{\underset{C^*(M \times M)}{\otimes}} C^*(M)[-\dim(M)]$$

$$\cong C^*(M^{S^n \vee S^n})[-\dim(M)].$$

Now, the commutativity of diagram (93) follows from Lemma 7.14. $\square$

7.3. A spectral sequence to compute the brane topology product. In [CJY], the Chas-Sullivan product of spheres and projective spaces was computed using a spectral sequence of algebras. There is a similar approach for the Brane product. Indeed, following arguments similar to those of [CJY], there exists a spectral sequence of algebras which converges (as algebras) to the homology $H_*((M)^{S^n})$ with product being the sphere product from (87). It is essentially given by the Serre spectral sequence applied to the fibration $\Omega^n M \to Map(S^n, M) \to M$.

Proposition 7.18. *Let M be closed oriented connected manifold. There exists a second quadrant spectral sequence of algebras $\{E^r_{p,q}, d^r : E^r_{p,q} \to E^r_{p-r,q+r-1}\}_{r\geq 1}$, which converges (as algebras) to the homology $H_{*+\dim(M)}((M)^{S^n})$ with product being the sphere product from (87).*

Furthermore, the E^2 page of the spectral sequence is given by

(94) $$E^2_{-p,q} \cong H^p(M; H_q(\Omega^n M)) \cong H^p(M) \otimes H_q(\Omega^n M).$$

Here, the algebra structure on $H^p(M)$ is given by the usual cup product on cohomology, and the product on $H_q(\Omega^n M)$ is the Pontrjagin product on the based loop space.

Proof. The spectral sequence is essentially given by the Serre spectral sequence applied to the fibration $\Omega^n M \to Map(S^n, M) \to M$. The construction is similar to the calculation of the loop product in [CJY, section 2] where we now consider the sphere product as defined by the sequence of maps in (87). We now give some of the details of this rather lengthy construction, and leave it to the interested reader to give the full details which are completely analogous to those in [CJY].

For a fibration $F \hookrightarrow E \overset{\pi}{\to} B$, consider the singular simplicial set with r-simplicies $S_r(E) = \{\sigma : \Delta^r \to E\}$. This simplicial set has a filtration obtained by taking simplicies whose induced simplicies on B are essentially of simplicial degree at most p,

$$F_p(S_r(E)) = \big\{\sigma : \Delta^r \to E : \pi \circ \sigma = \rho \circ f_\sharp, \text{ where } \rho \in S_q(B), q \leq p, \text{ and}$$

$$f_\sharp : \Delta^r \to \Delta^p \text{ is induced by some non-decreasing } f : \{0, \ldots, r\} \to \{0, \ldots, p\}\big\}$$

$F_p(S_\bullet(E))$ forms a subsimplicial set of $S_\bullet(E)$, and taking the associated chain complex, which is denoted by $F_p(C_*(E)) := C_*(F_p(S_\bullet(E)))$, induces the filtration

$$\{0\} \to \cdots \to F_{p-1}(C_*(E)) \to F_p(C_*(E)) \to \cdots \to C_*(E)$$

of $C_*(E)$. This is the filtration which in turn induces the Serre spectral sequence converging to $H_*(E)$.

Just as in [CJY], we may analyze the involved maps in the sphere product (87) and their behavior under the above filtration. For the space $(M^{S^n \vee S^n})^{-TM} = ev^*(M^{-TM})$ appearing in (87), which involves the Thom construction, we may in turn use the filtration of pairs,

$$F_p(C_*(ev^*(D_{TM}), ev^*(S_{TM}))) := F_p(C_*(ev^*(D_{TM})))/F_p(C_*(ev^*(S_{TM}))),$$

where $M^{-TM} = D_{TM}/S_{TM}$ is given by a collapsing the tubular disk D_{TM} of M in TM by the corresponding boundary sphere S_{TM}. This relative Serre spectral sequence converges to $H_*((M^{S^n \vee S^n})^{-TM})$. All the terms in (87) respect the filtration in the following sense:

$$F_p\big(C_*(M^{S^n})\big) \otimes F_q\big(C_*(M^{S^n})\big) \quad \longrightarrow \quad F_{p+q}\big(C_*(M^{S^n \amalg S^n})\big)$$

$$F_p\big(C_*(M^{S^n \amalg S^n})\big) \quad \overset{\text{collapse}_*}{\longrightarrow} \quad F_p(C_*(ev^*(D_{TM}), ev^*(S_{TM})))$$

$$F_p(C_*(ev^*(D_{TM}), ev^*(S_{TM}))) \quad \overset{\rho_{ev^*(th(M))}}{\longrightarrow} \quad F_{p-\dim(M)}(C_*(ev^*(D_{TM}), ev^*(S_{TM})))$$

$$F_p(C_*(ev^*(D_{TM}), ev^*(S_{TM}))) \quad \overset{\pi_*}{\longrightarrow} \quad F_p\big(C_*(M^{S^n \vee S^n})\big)$$

$$F_p\big(C_*(M^{S^n \vee S^n})\big) \quad \overset{(\delta_{S^n}^*)_*}{\longrightarrow} \quad F_p\big(C_*(M^{S^n})\big).$$

The above is obvious for most of the stated maps, except for the map $\rho_{ev^*(th(M))}$ involving capping with the Thom class, which can be proved just as in [CJY, Theorem 8]. Thus, we arrive at a map of filtered chain complexes

$$F_p(C_*(M^{S^n})) \otimes F_q(C_*(M^{S^n})) \to F_{p+q-\dim(M)}(C_*(M^{S^n}))$$

inducing a map of spectral sequences

$$\tilde{E}^r_{p,s} \otimes \tilde{E}^r_{q,t} \to \tilde{E}^r_{p+q-\dim(M),s+t}$$

such that on the $\tilde{E}^2$ page this is given by a map

$$H_p(M; H_s(\Omega^n M)) \otimes H_q(M; H_t(\Omega^n M)) \to H_{p+q-\dim(M)}(M; H_{s+t}(\Omega^n M)).$$

Checking the maps in the sphere product according to their component on the base M and the fiber $\Omega^n M$ shows that this product is just the intersection product on $H_*(M)$, and the Pontrjagin product on $H_*(\Omega^n M)$. Now, shifting the spectral sequence to the left by $\dim(M)$, we obtain another spectral sequence $E^r_{p,t} := \tilde{E}^r_{p+\dim(M),t}$, which now lives completely in the second quadrant, and for which the sphere product induces a map $E^r_{p,s} \otimes E^r_{q,t} \to E^r_{p+q,s+t}$ converging to the sphere product on homology,

$$H_{p+\dim(M)}(M^{S^n}) \otimes H_{q+\dim(M)}(M^{S^n}) \to H_{p+q+\dim(M)}(M^{S^n}).$$

The E^2 page of this new spectral sequence is given by $E^2_{p,s} = H_{p+\dim(M)}(M; H_s(\Omega^n M))$, so that, as a last step for obtaining the claimed spectral sequence in (94), we use Poincaré duality to write $E^2_{p,s} = H^{-p}(M; H_s(\Omega^n M))$. As the the intersection product becomes the cup product in cohomology under Poincaré duality, we finally arrive at the claimed spectral sequence. □

We end this section with a computation of the sphere product in the case where $M = S^k$ is itself a sphere of a certain dimension k.

Example 7.19. Let S^n denote the n-sphere, let $n > k$, and denote by $A = C^\bullet(S^k)$ a cochain model for the k-sphere. By formality of the sphere, we may assume that $A = H^\bullet(S^k)$. By Proposition 7.18, there exists a second quadrant spectral sequence of algebras $\{E^r_{p,q}, d^r : E^r_{p,q} \to E^r_{p-r,q+r-1}\}_{r \geq 1}$, which converges (as algebras) to the Brane homology $H_{*+k}((S^k)^{S^n})$; the product being the sphere product from (87). Furthermore, the E^2 page of the spectral sequence is given by

(95) $$E^2_{-p,q} \cong H^p(S^k; H_q(\Omega^n S^k)) \cong H^p(S^k) \otimes H_q(\Omega^n S^k).$$

Now we use the fact that $H^\bullet(S^k) = span\{1, a\}$ is a two dimensional space with a in degree k with the obvious cup product. To determine $H_\bullet(\Omega^n S^k)$, we distinguish between the cases where k is odd or even.

First, assume that $k = 2m + 1$ is odd. One calculates the homology with Pontrjagin product as $H_\bullet(\Omega^n S^k) \cong \Lambda(x)$ as the free algebra in one generator x in degree $k - n$. Thus, we can distinguish two cases, where n is even or odd. Figure 2 displays a picture of the generators of $H^\bullet(S^k) \otimes H_\bullet(\Omega^n S^k)$.

In the case where n is even, x is of odd degree $k - n$, so that $x^2 = 0$. Since the differential d^r on the rth page is of bi-degree $(-r, r - 1)$, all differentials have to be zero in this case. (More precisely, the rth differential would map bi-degree $(0, 0)$ to $(-r, r - 1) \neq (-k, k - n)$ for any r.)

In the case where n is odd, x is of even degree $k - n$, so that $x^2 \neq 0$. We see that the rth differential d^r could possibly nonzero, namely in the case where $(-r, r - 1) = (-k, p(k - n))$. Solving $r = k$ and thus $k - 1 = p(k - n)$, we see that the differential is always zero in the case when $k - 1$ is not a factor of $k - n$. (In the case where $k - 1$ is a factor of $k - n$ further analysis needs to be applied. For example, the above calculation applies when calculating the sphere product of $H_{*+13}((S^{13})^{S^5})$ but it would not apply to $H_{*+13}((S^{13})^{S^7})$ since $13 - 7 = 6$ is a factor of $13 - 1 = 12$.)
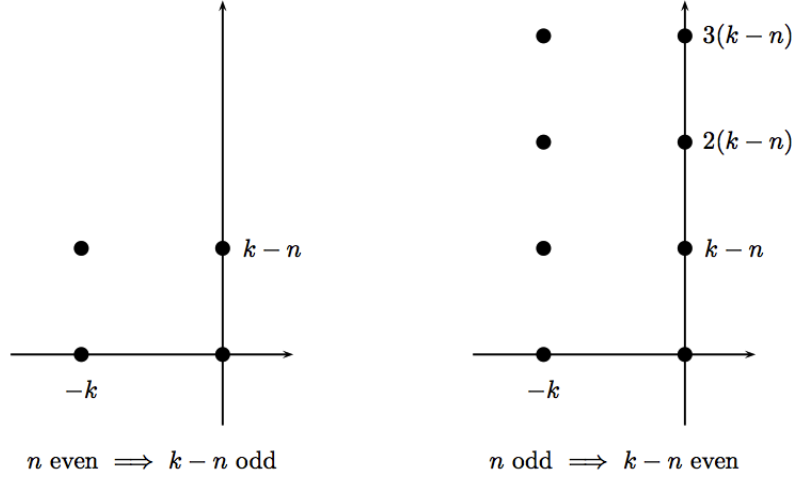
FIGURE 2. The generators of $H^\bullet(S^k) \otimes H_\bullet(\Omega^n S^k)$

We thus obtained the following result:

(96)   Let $k > n > 1$ with k odd.

If (n is even), or if (n is odd, and $k - n$ is not a factor of $k - 1$), then:

$$H_{*+k}\big((S^k)^{S^n}\big) = \Lambda[a] \otimes \Lambda[x], \qquad \text{with } |a| = -k \text{ and } |x| = k - n$$

One can ask for a similar analysis of the brane product when the underlying manifold is an even sphere S^k with $k = 2m$. One calculates the Pontrjagin ring as $H_\bullet(\Omega^n S^k) \cong \Lambda[x, y]$, where x is of degree $k - n$, and y is of degree $2k - 1 - n$. Unfortunately, the simple analysis of the involved degrees, similar to the one done for odd spheres, fails in this case, since one can check that the differential will never be zero purely by degree reasons. Thus, a computation of the brane product involves identifying the differentials of the spectral sequence, which are in general non trivial. Indeed, in [CJY], the differential d^k in the spectral sequence for k even and $n = 1$ was shown to be non-zero.

Remark 7.20. For an odd sphere S^k of sufficiently high connectivity, we have identified the sphere product with the cup product in Proposition 7.17, $C_*((S^k)^{S^n})[k] \cong CH^{S^n}(A, A)$, where we may take A to be the Sullivan model of the k-sphere. By Proposition 6.15, this structure is identified with the algebra structure on $CH^{S^n}(A, A) \cong HH_{\mathcal{E}_n}(A, A)$, which by Corollary 6.39, was also identified with $HH_{\mathcal{E}_n}(A, A) \cong \text{Sym}_A(\text{Der}(A, A)[n])$. This, in turn, was explicitly calculated in Example 6.40. Note that the two calculations in Examples 7.19 and 6.40 for the product on $C_*((S^k)^{S^n})[k]$ do indeed produce the same result.

8. ITERATED BAR CONSTRUCTIONS

8.1. Iterated loop spaces and iterated bar constructions for E_∞-algebras.
In this section we study the case of spaces of pointed maps from spheres to X, *i.e.* iterated loop spaces. The idea is to apply the formalism of the higher Hochschild functor to the Bar construction of augmented E_∞-algebras.

Let (A, d) be a differential graded unital associative algebra (DGA for short) which is equipped with an augmentation $\epsilon : A \to k$. Denote $\overline{A} = ker(A \xrightarrow{\epsilon} k)$ the augmentation ideal of A. The standard bar construction on A is the chain complex $(Bar^{std}(A), b)$ defined by

$$Bar^{std}(A) = \bigoplus_{n \geq 1} \overline{A}^{\otimes n}$$

with differential given by

$$
\begin{aligned}
b(a_1 \otimes \cdots a_n) &= \sum_{i=1}^{n} \pm a_1 \otimes \cdots \otimes d(a_i) \otimes \cdots \otimes a_n \\
&+ \sum_{i=1}^{n-1} \pm a_1 \otimes \cdots \otimes (a_i \cdot a_{i+1}) \otimes \cdots \otimes a_n
\end{aligned}
$$

see [FHT, Fre2, KM] for details (and signs). Further, if A is a commutative differential graded algebra (CDGA for short), then the shuffle product makes the Bar construction $Bar^{std}(A)$ a CDGA as well.

Remark 8.1. Note that, by our convention on $\otimes_k$, if A is not flat over k, we replace it by a flat resolution. In particular $Bar^{std} : E_1 - Alg \to k\text{-Mod}_\infty$ preserves weak equivalences. This definition of $Bar^{std}(A)$ thus agrees with the classical one as soon as the underlying chain complex of A is flat over k.

Remark 8.2. The standard bar construction above extends naturally to A_∞-algebras. It also extends to any augmented E_1-algebra. Indeed, one can prove a Lemma similar to Lemma 8.3 below with factorization homology $\int_I (A, k)$ (of the E_1-algebra) instead of Hochschild chains over I (see [F1, F2, L-HA, AFT]). Note that, there is a natural equivalence $\int_I (A, k) \cong k \overset{\mathbb{L}}{\underset{A}{\otimes}} k$. In particular, if B is any DGA equivalent to A, then the standard bar construction of A is naturally equivalent to $Bar^{std}(B)$ in $k\text{-Mod}_\infty$.

Let A be an E_∞-algebra and let $\epsilon : A \to k$ be an augmentation. In particular, we can see k as an A-module thanks to the augmentation ϵ. In particular A is an E_1-algebra so that we can choose a DGA B and a quasi-isomorphism $f : B \to A$ of E_1-algebras. Then we define $Bar^{std}(A) := Bar^{std}(B)$. The fact that this construction is well-defined[45] indeed follows from the following lemma:

Lemma 8.3. *Let $I = [0, 1]$ be the closed interval. There is a natural equivalence (in k-Mod_∞)*

$$(97) \qquad\qquad CH_I(A) \underset{CH_{S^0}(A)}{\overset{\mathbb{L}}{\otimes}} k \; \cong \; Bar^{std}(A)$$

where $Bar^{std}(A)$ is the standard Bar construction. Further if A is a CDGA, the equivalence (97) is an equivalence of E_∞-algebras (where $Bar^{std}(A)$ is endowed with its CDGA-structure induced by the shuffle product).

Proof. Let $I_\bullet^{std}$ be the standard simplicial set model of the interval (viewed as a CW-complex with two vertices and one non-degenerate 1-cell). More precisely, $I_k^{std} = \{0, \dots, k+1\}$ with face maps d_i given, for $i = 0, \dots, k$, by $d_i(j)$ equal to j

---

[45]i.e. independent of the choice of B

or $j-1$ depending on $j \leq i$ or $j > i$. For any differential graded associative algebra B, one can form the simplicial dg-algebra

$$\widetilde{C_{I_{\bullet}^{std}}}(B) := (B^{\otimes I_k^{std}})_{k \geq 0} = (B \otimes B^k \otimes B)_{k \geq 0}$$

where the simplicial structure is defined as for Hochschild chains of a CDGA (it is immediate to check, and well known, that the commutativity is not necessary to check the simplicial identities in that case). Further the associated[46] differential graded module $DK(\widetilde{C_{I_{\bullet}^{std}}}(B))$ is the two-sided Bar construction $Bar^{std}(B,B,B)$ of B (see [GTZ, Example 2.3.4]). In particular, if $f : B \xrightarrow{\simeq} A$ is an equivalence of E_1-algebras, with B a DGA, then $DK(\widetilde{C_{I_{\bullet}^{std}}}(B)) \xrightarrow{f_*} Bar^{std}(A)$ is an equivalence (natural in A, B).

Now, forgetting the E_{∞}-structure of the Hochschild chain complex $CH_{I_{\bullet}^{std}}(A)$, we get a quasi-isomorphism of simplicial chain complexes

$$f : \widetilde{C_{I_{\bullet}^{std}}}(B) \xrightarrow{\simeq} \widetilde{CH_{I_{\bullet}^{std}}}(A)$$

and thus after taking the Dold-Kan ∞-functor $DK : sk\text{-}Mod_{\infty} \to k\text{-}Mod_{\infty}$, we see that $CH_{I_{\bullet}^{std}}(A) \cong DK(\widetilde{C_{I_{\bullet}^{std}}}(B))$. Now the result follows since $Bar^{std}(B) \cong \cong \bigoplus_{n \geq 1} \overline{B}^{\otimes n}$ is the normalized chain complex associated to the simplicial chain complex $\widetilde{C_{I_{\bullet}^{std}}}(B)$, thus is quasi-isomorphic to $DK(\widetilde{C_{I_{\bullet}^{std}}}(B))$.

When A is a CDGA, the result follows from Corollary 3.7 and [GTZ, Section 2]. □

In particular, we get an E_{∞}-lifting of the Bar construction of an E_{∞}-algebra that we denote

$$(98) \qquad Bar(A) := CH_I(A) \overset{\mathbb{L}}{\underset{CH_{S^0}(A)}{\otimes}} k.$$

Note that the augmentation $\epsilon : A \to k$ induces augmentations $\epsilon_* : CH_I(A) \to CH_I(k) \cong k$, $\epsilon_* : CH_{S^0}(A) \to CH_{S^0}(k) \cong k$ and thus an augmentation $Bar(A) \to k$ as well.

Since $Bar(A)$ is an augmented E_{∞}-algebra, we can take its Bar construction again.

Definition 8.4. The nth-iterated Bar construction of an augmented E_{∞}-algebra A is the E_{∞}-algebra $Bar^{(n)}(A) = Bar(\cdots(Bar(A))\cdots)$.

Summing up the above results we have:

Proposition 8.5. *The nth-iterated Bar construction Bar is an ∞-functor*

$$Bar^{(n)} : E_{\infty}\text{-}Alg \to E_{\infty}\text{-}Alg.$$

Further, there is a natural equivalence in E_{∞}-Alg between $B^{(n)}(A)$ and the nth-iterated Bar construction defined by B. Fresse [Fre2].

Proof. Since $A \mapsto CH_I(A)$ is an ∞-endofunctor of E_{∞}-Alg, the same follows for Bar (and its iteration). By Lemma 8.3, the $Bar(A)$ is equivalent (in $k\text{-}Mod_{\infty}$) to $Bar^{std}(A)$ and, further, this equivalence is an equivalence in E_{∞}-Alg if A is a CDGA and $Bar^{std}(A)$ is endowed with the CDGA structure given by the shuffle

---

[46] via the usual Dold-Kan construction

product. Thus the uniqueness of the Bar construction in E_∞-Alg obtained in [Fre2] shows that $Bar^{(n)}$ is the correct n^{th}-iterated Bar construction. $\qquad\square$

Remark 8.6. Since the canonical map $CH_I(A) \to CH_{pt}(A) \cong A$ is an equivalence, we recover immediately from the excision axiom

$$Bar(A) \cong A \overset{\mathbb{L}}{\underset{CH_{S^0}(A)}{\otimes}} k \cong k \overset{\mathbb{L}}{\underset{A}{\otimes}} k.$$

Remark 8.7. In terms of factorization algebras, one has the following definition. Considered the unit interval with two stratified points given by its endpoints. Then, the analogue of Proposition 5.6 in that case is that a locally constant (stratified) factorization algebra on I is the same as the data of an E_1-algebra A and a pair of left A-module M and a right A-module N. In particular taking the factorization algebra $\mathcal{A}$ for which A is augmented and $M = N = k$, we obtain that the factorization homology $\int_I \mathcal{A}$ (denoted $\int_I(A, k)$ in [F1]) is equivalent to the Bar construction, see [F1] for details.

There is an easy interpretation of the iterated Bar construction in terms of higher Hochschild chains. Note that, since k is an A-algebra (via the augmentation), $CH_{S^n}(A, k) \cong CH_{S^n}(A) \overset{\mathbb{L}}{\underset{A}{\otimes}} k$ is an E_∞-algebra.

Proposition 8.8. *There are natural equivalences of E_∞-algebras*

$$CH_{S^n}(A, k) \cong Bar^{(n)}(A).$$

Proof. Since $S^n \cong D^n \cup^h_{S^{n-1}} pt$, the homotopy invariance and excision axiom for Hochschild chains implies the following sequence of natural (in A) equivalences of E_∞-algebras

$$CH_{S^n}(A, k) \;\cong\; CH_{S^n}(A) \overset{\mathbb{L}}{\underset{A}{\otimes}} k \;\cong\; CH_{I^n}(A) \overset{\mathbb{L}}{\underset{CH_{S^{n-1}}(A)}{\otimes}} k$$

Thus, for $n = 1$, the Lemma is proved (by Definition (98)).

Since $CH_X(k) \cong k$ for all $X \in Top_\infty$, by Corollary 3.29.(3), there are equivalences of E_∞-algebras

$$
\begin{aligned}
CH_I\Big(CH_{S^{n-1}}(A) \overset{\mathbb{L}}{\underset{A}{\otimes}} k\Big) \;&\cong\; CH_I\big(CH_{S^{n-1}}(A)\big) \overset{\mathbb{L}}{\underset{CH_I(A)}{\otimes}} k \\
&\cong\; CH_{(I \times S^{n-1})/I \times \{1\}}(A).
\end{aligned}
$$

where the last equivalence follows from Corollary 3.29.(4) and the excision axiom. Tensoring the last equivalence by $\overset{\mathbb{L}}{\underset{CH_{S^0}\big(CH_{S^{n-1}}(A,k)\big)}{\otimes}} k$ and applying the excision axiom again, we get

$$CH_I\Big(CH_{S^{n-1}}(A) \overset{\mathbb{L}}{\underset{A}{\otimes}} k\Big) \overset{\mathbb{L}}{\underset{CH_{S^0}\big(CH_{S^{n-1}}(A,k)\big)}{\otimes}} k \;\cong\; CH_{S^n}(A) \overset{\mathbb{L}}{\underset{A}{\otimes}} k.$$

Since the left hand side is $Bar\Big(CH_{S^{n-1}}(A) \overset{\mathbb{L}}{\underset{A}{\otimes}} k\Big)$, the Lemma now follows by induction. $\qquad\square$

We now study the coalgebra structure carried by the iterated Bar construction. Recall that the standard Bar construction of a DGA carries a natural associative coalgebra structure. We wish to apply the results of Section 4.2 to study the same result for E_n-coalgebras structures.

Recall the continuous map (25) $pinch : \mathcal{C}_n(r) \times S^n \longrightarrow \bigvee_{i=1\dots r} S^n$. Similarly to the definition of the map (26), applying the singular set functor to the map (25) we get a morphism

$$(99) \quad pinch_*^{S^n,r} : C_* \big(\mathcal{C}_n(r) \big) \otimes CH_{S^n}(A) \overset{\mathbb{L}}{\underset{A}{\otimes}} k$$

$$\overset{pinch_* \otimes_A^{\mathbb{L}} id}{\longrightarrow} CH_{\bigvee_{i=1}^r S^n}(A) \overset{\mathbb{L}}{\underset{A}{\otimes}} k \cong \Big(CH_{\coprod_{i=1}^r S^n}(A) \overset{\mathbb{L}}{\underset{A^{\otimes r}}{\otimes}} A \Big) \overset{\mathbb{L}}{\underset{A}{\otimes}} k$$

$$\cong \Big(CH_{\coprod_{i=1}^r S^n}(A) \Big) \overset{\mathbb{L}}{\underset{A^{\otimes r}}{\otimes}} k \cong \Big(CH_{S^n}(A,k) \Big)^{\otimes r}$$

where the last equivalences follows from the excision axiom, the coproduct axiom and the definition of $CH_{S^n}(A,k)$.

Note that there is a canonical equivalence

$$(100) \quad Hom_k \Big(CH_{S^n}(A) \otimes_A^{\mathbb{L}} k, k \Big) \cong RHom_A \Big(CH_{S^n}(A), k \Big) \cong CH^{S^n}(A,k).$$

Under this identification, the dual of the map (99) is the pinching map (26) from Section 4.1.

Theorem 8.9. *Let A be an E_∞-algebra and $\epsilon : A \to k$ an augmentation.*

(1) *The maps (99) $pinch_*^{S^n,r} : C_* \big(\mathcal{C}_n(r) \big) \otimes CH_{S^n}(A,k) \to \Big(CH_{S^n}(A,k) \Big)^{\otimes r}$ makes the iterated Bar construction $Bar^{(n)}(A) \cong CH_{S^n}(A,k)$ a natural E_n-coalgebra (in the $(\infty, 1)$-category of E_∞-algebras)*

(2) *The dual E_n-algebra $RHom(Bar^{(n)}(A), k)$ is naturally equivalent to $CH^{S^n}(A,k)$ in E_n-Alg and thus to the centralizer $\mathfrak{z}(\epsilon)$ of the augmentation (viewed as a map of E_n-algebra by restriction).*

Proof. The proof of the first statement is similar to the proof of Theorem 4.12 (except that we take the predual of it). Fixing $c \in C_*(\mathcal{C}_n(r))$, all maps involved in the composition (99) defining $pinch_*^{S^n,r}(c, -)$ are maps of E_∞-algebras. Hence the structure maps of the E_n-coalgebra structures are compatible with the E_∞-structure.

Further, since the linear dual of the map (99) is the pinching map (26), statement (2) follows from Theorem 4.12, the equivalence

$$RHom(Bar^{(n)}(A), k) \cong RHom_A \Big(CH_{S^n}(A), k \Big) \cong CH^{S^n}(A,k)$$

and Corollary 6.26 $\qquad \square$

If Y is a pointed space, its E_∞-algebra of cochains $C^*(Y)$ has a canonical augmentation $C^*(Y) \to C^*(pt) \cong k$ induced by the base point $pt \to Y$. Tensoring the map $\mathcal{I}t : CH_{S^n}(C^*(Y)) \longrightarrow C^* \big(Y^{S^n} \big)$ (given by Theorem 3.33) with $\otimes_{C^*(Y)}^{\mathbb{L}} k$ yields a natural E_∞-algebra morphism

$$(101) \quad \mathcal{I}t^{\Omega^n} : Bar^{(n)}(C^*(Y)) \cong CH_{S^n}(C^*(Y), k)$$

$$\overset{\mathcal{I}t \otimes_{C^*(Y)}^{\mathbb{L}} k}{\longrightarrow} C^* \big(Y^{S^n} \big) \otimes_{C^*(Y)}^{\mathbb{L}} k \longrightarrow C^* \big(\Omega^n(Y) \big)$$

where the last map is induced by applying the singular cochain functor to $\Omega^n(Y) \cong Y^{S^n} \times_Y^h pt$.

Further, using the equivalence (100), the linear dual of this map (composed with the canonical biduality morphism) yields a map

(102)
$$\mathcal{I}t_{\Omega^n} : C_*\big(\Omega^n(Y)\big) \longrightarrow C^*\big(\Omega^n(Y)\big)^\vee \longrightarrow CH^{S^n}(C^*(Y), k) \cong \Big(Bar^{(n)}(C^*(Y))\Big)^\vee$$

We can now state our main application to iterated loop spaces, generalizing classical results in algebraic topology. Since the iterated loop space $\Omega^n(Y)$ are E_n-algebras in spaces, $C^*(\Omega^n(Y))$ is an $E_n - coalgebra$ in E_∞-Alg and $C_*(\Omega^n(Y))$ an E_n-algebra (in E_∞-coAlg, the $(\infty, 1)$-category of E_∞-coalgebras).

Corollary 8.10. *Let Y be a pointed topological space.*

(1) *The map (101) $\mathcal{I}t^{\Omega^n} : Bar^{(n)}(C^*(Y)) \to C^*\big(\Omega^n(Y)\big)$ is an E_n-coalgebra morphism in the category of E_∞-algebras. It is further an equivalence if Y is n-connected.*

(2) *Dually, the map (102) $\mathcal{I}t_{\Omega^n} : C_*\big(\Omega^n(Y)\big) \longrightarrow \Big(Bar^{(n)}(C^*(Y))\Big)^\vee$ is an E_n-algebra morphism (in k-Mod$_\infty$). Further, if k is a field, Y is n-connected and has finite dimensional homology groups, then $\Big(Bar^{(n)}(C^*(Y))\Big)^\vee$ is an E_∞-coalgebra and the map (102) $\mathcal{I}t_{\Omega^n}$ is an equivalence of E_n-algebras in E_∞-coAlg.*

In particular, the Hochschild chains over the spheres is a model for the natural E_n-algebra structure on $C_*(\Omega^n Y)$.

Proof. By Theorem 3.33, the map $\mathcal{I}t : CH_{S^n}(C^*(Y)) \longrightarrow C^*\big(Y^{S^n}\big)$ is an E_∞-algebra map and thus so is $\mathcal{I}t_{\Omega^n}$. Further, Theorem 3.33 gives a natural transformation
$$\mathcal{I}t : CH_X^*\big(C^*(Y)\big) \longrightarrow C^*\big(Y^X\big)$$
from which we deduce a commutative diagram

(103)
$$CH_{I^n}\big(C^*(Y)\big) \overset{\mathbb{L}}{\underset{CH_{S^{n-1}}\big(C^*(Y)\big)}{\otimes}} C^*(Y) \overset{\mathbb{L}}{\underset{C^*(Y)}{\otimes}} k \overset{\simeq}{\longrightarrow} CH_{S^n}\big(C^*(Y)\big) \overset{\mathbb{L}}{\underset{C^*(Y)}{\otimes}} k$$

$$\Big\downarrow {\scriptstyle \mathcal{I}t \otimes^{\mathbb{L}}_{\mathcal{I}t} id} \qquad\qquad\qquad\qquad \Big\downarrow {\scriptstyle \mathcal{I}t \otimes^{\mathbb{L}}_{C^*(Y)} id}$$

$$C^*\big(Y^{I^n}\big) \overset{\mathbb{L}}{\underset{C^*\big(Y^{S^{n-1}}\big)}{\otimes}} C^*(Y) \overset{\mathbb{L}}{\underset{C^*(Y)}{\otimes}} k \longrightarrow C^*\big(Y^{S^n}\big) \overset{\mathbb{L}}{\underset{C^*(Y)}{\otimes}} k$$

in E_∞-Alg in which the horizontal arrows are induced by the homotopy pushout $\Omega^n X \cong X^{I^n} \cup_{X^{S^{n-1}}} pt$. The lower horizontal arrow is an equivalence when Y is n-connected. Further, the map $\mathcal{I}t : CH_{S^{n-1}}\big(C^*(Y)\big) \to C^*\big(Y^{S^{n-1}}\big)$ is an equivalence when Y is $n-1$-connected by Theorem 3.33. Since the map induced by the base point $C^*(Y) \to CH_{I^n}(C^*(Y))$ is an equivalence, the map $\mathcal{I}t : CH_{I^n}(C^*(Y)) \to C^*\big(Y^{I^n}\big)$ is an equivalence when Y is connected. Thus, we deduce from the commutativity of diagram (103) that the map $\mathcal{I}t^{\Omega^n} : Bar^{(n)}(C^*(Y)) \to C^*\big(\Omega^n(Y)\big)$ is an equivalence when Y is n-connected.

In order to finish the proof of Assertion 1 in Corollary 8.10, it remains to check that $\mathcal{I}t^{\Omega^n}$ is a map of E_n-coalgebras. By definition, the E_n-coalgebra structure

of $C^*\big(\Omega^n(Y)\big)$ is induced by taking the singular cochains functor (from Top_∞ to E_∞-Alg) to the E_n-algebra structure of $\Omega^n(Y)$ which is the (homotopy pullback) $\Omega^n(Y) \cong \big(Y^{S^n} \times_Y pt\big)$. By definition the E_n-algebra structure of $\Omega^n(Y)$ is induced by the pinching map (26) $\mathcal{C}_n(r) \times S^n \to \bigvee_{i=1\dots r} S^n$. Indeed, since the pinching map preserves the base point of S^n, we have the following composition

$$(104) \quad \mathcal{C}_n(r) \times \big(Y^{S^n} \times_Y^h pt\big)^r \xrightarrow{\cong} \mathcal{C}_n(r) \times \big(Y^{\coprod_{i=1\dots r} S^n}\big) \times_{Y^r} pt$$

$$\xrightarrow{\cong} \big(Y^{\vee_{i=1\dots r} S^n}\big) \times_Y pt \xrightarrow{pinch^*} Y^{S^n} \times_Y pt.$$

By naturality of $\mathcal{I}t$, we have a commutative diagram

$$CH_{S^n}\big(C^*(Y)\big) \overset{\mathbb{L}}{\underset{C^*(Y)}{\otimes}} k \xrightarrow{\;pinch_* \overset{\mathbb{L}}{\underset{C^*(Y)}{\otimes}} id\;} CH_{\vee_{i=1\dots r} S^n}\big(C^*(Y)\big) \overset{\mathbb{L}}{\underset{C^*(Y)}{\otimes}} k \;.$$

$$\downarrow {\scriptstyle \mathcal{I}t \otimes_{C^*(Y)}^{\mathbb{L}} id} \qquad\qquad\qquad\qquad\qquad \downarrow {\scriptstyle \mathcal{I}t \otimes_{C^*(Y)}^{\mathbb{L}} id}$$

$$C^*\big(Y^{S^n}\big) \overset{\mathbb{L}}{\underset{C^*(Y)}{\otimes}} k \xrightarrow{\;C^*(pinch^*)\;} C^*\big(Y^{\vee_{i=1\dots r} S^n}\big) \overset{\mathbb{L}}{\underset{C^*(Y)}{\otimes}} k$$

The commutativity of this diagram, together with the definition of the map (99) $pinch_*^{S^n,r} : C_*\big(\mathcal{C}_n(r)\big) \otimes CH_{S^n}(A) \overset{\mathbb{L}}{\underset{A}{\otimes}} k \to \big(CH_{S^n}(A) \overset{\mathbb{L}}{\underset{A}{\otimes}} k\big)^{\otimes r}$ giving the E_n-coalgebra structure of $Bar^{(n)}(C^*(Y))$, and the fact that the E_n-coalgeba structure of $C^*(\Omega^n(Y))$ is given by applying the functor $C^*(-)$ to the composition (104) show that $\mathcal{I}t^{\Omega^n}$ is an E_n-algebra map.

The proof of the fact that $\mathcal{I}t_{\Omega^n}$ is a map of E_n-algebra is similar, using in addition the naturality of the biduality morphism $C \to C^{\vee\vee}$ and Corollary 3.36. Further, when k is a field and the groups $H_\ell(Y)$ are finitely generated, then $C_*(Y) \to (C^*(Y))^\vee$ is an equivalence. Further, if Y is n-connected, it follows from the Eilenberg-Moore spectral sequence that $Bar^{(n)}(C^*(Y))$ has finite dimensional homology groups. Hence, the dual $\big(Bar^{(n)}(C^*(Y))\big)^\vee$ inherits an natural E_∞-coalgebra structure (dual of the E_∞-algebra structure of $Bar^{(n)}(C^*(Y))$). It is then immediate to check that the arguments to prove Statement (1) above can be dualized to prove that $\mathcal{I}t_{\Omega^n}$ is also an equivalence of E_∞-coalgebras. $\qquad\square$

Remark 8.11. A careful analysis of the proof of Corollary 8.10 shows that the assumption that Y is n-connected can be replaced by the assumption that the cohomological Eilenberg-Moore spectral sequence of the path space fibration is strongly convergent for all $\Omega^i Y$ $(i \le n)$.

Remark 8.12. Statement (2) in Corollary 8.10 is somehow unsatisfying since one recovers an E_∞-coalgebra structure on the right hand side $\big(Bar^{(n)}(C^*(Y))\big)^\vee$ only when the biduality morphism $Bar^{(n)}(C^*(Y)) \to \big(Bar^{(n)}(C^*(Y))\big)^{\vee\vee}$ is an equivalence (while the left hand side has always such a structure). The reason for it is that this statement is in fact the bidual of a statement involving iterated *coBar* construction of E_∞-coalgebras.

Indeed, one can define Hochschild cochains over spaces for E_∞-coalgebras in a similar way to what we do in Section 3 getting an ∞-functor $CH : Top_\infty^{op} \times$

E_∞-coAlg $\to$ E_∞-coAlg $((X, C) \mapsto CH^X(C))$. For instance, one has a natural equivalence $CH^X(C) \cong C \overset{\mathbb{L}}{\underset{\mathbb{E}_\infty^\otimes}{\otimes}} C^*(X)$ similar to Proposition 3.6.

All results of Section 3, Section 3.4 and Section 4 have "dual" counterparts which can be proved similarly. We claim that there is an iterated cobar construction $coBar^{(n)} : E_\infty$-coAlg $\to E_n$-Alg$(E_\infty$-coAlg$)$ defined similarly to this Section 8.1 and that further there is a natural E_n-algebra map $coBar^{(n)}(C_*(Y)) \to C_*(\Omega^n(Y))$ in E_∞-coAlg which is an equivalence when Y is n-connected. We leave the many details to be filled in to the interested reader.

8.2. Iterated Bar constructions of augmented E_n-algebras.
In this section we explain how to generalize the iterated Bar construction for E_∞-algebras in § 8.1 to E_n-algebras. In particular we describe the E_n-coalgebra structure of the n-times iterated Bar construction. Our definition and study of the Bar construction follows the ones given by Francis [F1] and Lurie [L-HA].

8.2.1. *Definition of the iterated Bar construction for E_n-algebras.* In this section we assume A is an augmented E_n-algebra and we denote $\epsilon : A \to k$ the augmentation (which is a map of E_n-algebras). In particular, we endow k with its structure of A-E_n-module given by the augmentation. We denote E_n-Alg^{aug} *the $(\infty, 1)$-category of augmented E_n-algebras.*

For an augmented E_n-algebra, Definition (98) and Lemma 8.3 suggest to define

$$(105) \qquad Bar(A) := \int_{D^1 \times \mathbb{R}^{n-1}} A \overset{\mathbb{L}}{\underset{\int_{S^0 \times \mathbb{R}^{n-1}} A}{\otimes}} k$$

where $k \cong \int_{I \times \mathbb{R}^{n-1}} k$ is endowed with its natural structure of A-E_1-module. This definition agrees with the usual one:

Lemma 8.13 (Francis [F1]). *There is a natural equivalence (in k-Mod_∞)*

$$(106) \qquad Bar(A) \cong Bar^{std}(A) \cong k \otimes_A^{\mathbb{L}} k$$

where $Bar^{std}(A)$ is the standard Bar construction as in § 8.1.

When X be a manifold of dimension d equipped with a framing of $X \times \mathbb{R}^k$, then for any E_{d+k}-algebra B, $\int_{X \times \mathbb{R}^k} B$ is canonically an E_k-algebra, see [L-HA, F1] for details. Note that this follows from Theorem 2.20 and the fact that factorization algebras on $X \times \mathbb{R}^k$ are the same as factorization algebras on X with values in E_d-Alg (see Theorem 2.29 or [GTZ2]). Applying this observation to $X = I$ or $X = S^0$ we get the following result which is also proved in [F1, L-HA].

Proposition 8.14. *The Bar construction* (106) *for augmented E_m-algebras ($m \geq 1$) has a canonical lift*

$$Bar : E_m\text{-}Alg^{aug} \to E_{m-1}\text{-}Alg^{aug}$$

which coincides for E_∞-algebras with the one given in § 8.1 and further sits into a commutative diagram

$$
\begin{array}{ccccccccc}
E_1\text{-}Alg & \longleftarrow & E_2\text{-}Alg & \longleftarrow & \cdots & \longleftarrow & E_m\text{-}Alg & \longleftarrow \cdots \longleftarrow & E_\infty\text{-}Alg \\
\downarrow{\scriptstyle Bar} & & \downarrow{\scriptstyle Bar} & & & & \downarrow{\scriptstyle Bar} & & \downarrow{\scriptstyle Bar} \\
k\text{-}Mod_\infty & \longleftarrow & E_1\text{-}Alg & \longleftarrow & \cdots & \longleftarrow & E_{m-1}\text{-}Alg & \longleftarrow \cdots \longleftarrow & E_\infty\text{-}Alg
\end{array}
$$

where the horizontal arrows are the canonical forget functors induced by the tower of maps of operads (2).

Proof. By Theorem 2.20 (and the above observation which is a special case of of the Fubini formula for factorization homology [GTZ2, Corollary 17]), we have that $\int_{S^0\times\mathbb{R}^{m-1}} A$, $\int_{I\times\mathbb{R}^{m-1}} A$ and $k \cong \int_{I\times\mathbb{R}^{m-1}} A$ are (global sections of) locally constant factorizations algebras over $\mathbb{R}^{m-1}$. Here, we see S^0 as being the boundary of the closed interval $I = [-1,1]$ which is framed (we choose the framing so that the induced orientation is the natural one); in particular $S^0 \cong \{-1,1\}$ inherits a framing as well (note that the two points in S^0 get opposite orientation this way). In particular $S^0\times\mathbb{R}^{m-1}$ is equipped with the product framing. Since $S^0\times\mathbb{R}^{m-1}$ is a framed $m-1$ dimensional manifold and A an E_m-algebra, its factorization homology with value in A is the one of the product of framed manifolds $S^0 \times \mathbb{R}^{m-1} \times \mathbb{R}$. Hence, we have that $\int_{S^0\times\mathbb{R}^{m-1}} A$ is in fact an E_m-algebra, that is a locally constant factorization algebra over $\mathbb{R}^{m-1} \times \mathbb{R}$.

In particular, using Theorem 2.30, $\int_{S^0\times\mathbb{R}^{m-1}} A$ is naturally an E_1-algebra in the symmetric monoidal category of E_{m-1}-algebras, *i.e.*, an E_1-algebra in the category of locally constant factorizations algebras over $\mathbb{R}^{m-1}$.

Similarly $\int_{I\times\mathbb{R}^{m-1}} A$ is a left module over $\int_{S^0\times\mathbb{R}^{m-1}} A$ in the symmetric monoidal category locally constant factorizations algebras over $\mathbb{R}^{m-1}$ (or equivalently of E_{m-1}-algebras). In other words, it belongs to $\left(\int_{S^0\times\mathbb{R}^{m-1}} A\right)$-$Mod^{E_1}\left(\mathrm{Fac}^{lc}_{\mathbb{R}^{m-1}}\right)$ which is equivalent to $\left(\int_{S^0\times\mathbb{R}^{m-1}} A\right)$-$Mod^{E_1}\left(E_{m-1}\text{-Alg}\right)$. Since the same holds for k, we obtain that the Bar construction is an object in $\mathrm{Fac}^{lc}_{\mathbb{R}^{m-1}}$, hence inherits a structure of E_{m-1}-algebra.

Further, the augmentation $\epsilon : A \to k$ induces a maps $\int_{I\times\mathbb{R}^{m-1}} \epsilon : \int_{I\times\mathbb{R}^{m-1}} A \to k$ which is a map of locally constant factorization algebras on $\mathbb{R}^{m-1}$ hence of E_{m-1}-algebras. Similarly $\int_{S^0\times\mathbb{R}^{m-1}} A \to k$ is a map of E_m-algebras; hence ϵ induces an augmentation $Bar(A) \to k$ in E_{m-1}-Alg. The equivalence of the two definitions for E_∞-algebras is an immediate consequence of Theorem 3.13 or [GTZ2, Theorem 5].

The commutativity of the diagram follows from the fact that E_m-Alg $\to E_{m-1}$-Alg is induced by the map of ∞-operad $\mathbb{E}^\otimes_{m-1} \to \mathbb{E}^\otimes_m$ induced by taking the product of $m-1$-dimensional disks with the interval $\mathbb{R}$, *i.e*, it is induced by the pushforward of factorization algebras along the projection $\mathbb{R}^{m-1} \times \mathbb{R} \to \mathbb{R}^{m-1}$. □

By Proposition 8.14, we can iterate (up to m-times) the Bar constructions of an E_m-algebra.

Definition 8.15. Let $0 \leq n \leq m$. The nth-iterated bar construction of an augmented E_m-algebra A is the E_{m-n}-algebra (given by Proposition 8.14)

$$Bar^{(n)}(A) := Bar(\cdots(Bar(A))\cdots)$$

which is the value on A of the (n-fold iterated Bar) functor: $Bar^{\circ n} : E_m\text{-Alg}^{aug} \longrightarrow E_{m-n}\text{-Alg}^{aug}$.

Proposition 8.14 implies that Definition 8.15 agrees with Definition 8.4 for E_∞-algebras.

Remark 8.16. The iterated Bar construction given in Definition 8.15 should be closely related to the one (obtained at the level of model categories) by Fresse [Fre3].

The following result, due to Francis [F1, Lemma 2.44], identifies the iterated Bar construction in terms of factorization homology

Lemma 8.17 (Francis)**.** *Let A be an E_m-algebra and $0 \leq n \leq m$. There is a natural equivalence of E_{m-n}-algebras*

$$Bar^{(n)}(A) \cong \int_{D^m \times \mathbb{R}^{m-n}} A \overset{\mathbb{L}}{\underset{\int_{S^{m-1} \times \mathbb{R}^{m-n+1}} A}{\otimes}} k$$

Proof. This is essentially Lemma 2.44 together with Corollary 3.32 in [F1]. Alternatively, on can use a proof similar to the one of Proposition 8.8 replacing $CH_{I^n}(A)$ with the E_{m-n}-algebra $\int_{I^m \times \mathbb{R}^{m-n}} A$ using excision for factorization homology (see [F1, AFT, GTZ2]), and the Fubini theorem for factorization homology [GTZ2, Corollary 17] instead of Corollary 3.29.(4). □

8.2.2. *Factorization algebra models for iterated Bar construction.* In this section, we show that the iterated Bar construction can be computed as factorization homology of a stratified factorization algebra on the closed n-disk and the n-sphere as well.

Identify $I^n = [-1,1]^n$ with the closed unit disk in $\mathbb{R}^n$ and let $D^n = I^n \setminus \partial I_n$ be its interior. We consider I^n as a stratified space with two strata, one of which is its boundary ∂I^n (of codimension 1) and the remaining open strata being D^n. A factorization algebra $\mathcal{F}$ on the stratified disk is thus locally constant if for any inclusion of disks $U \subset V \subset D^n$ and for any inclusion of half-disks $U \subset V$ where $V \not\subset D^n$, the structure map $\mathcal{F}(U) \to \mathcal{F}(V)$ is a quasi-isomorphism.

Let $\epsilon : A \to k$ be an augmented E_n-algebra which we may assume to be given by a map $\epsilon : \mathcal{A} \to k$ of factorization algebras. By [G2, Proposition 30 and Proposition 31][47], any map of E_n-algebra $f : A \to B$ defines a locally constant stratified factorization algebra on I^n. Indeed, by *loc. cit.* we have a faithful functor

(107) $$\Upsilon : Hom_{E_n\text{-Alg}} \longrightarrow \text{Fac}_{I^n}^{lc}$$

between the $(\infty, 1)$-categories of E_n-algebras morphisms and stratified locally constant factorization algebras on I^n.

Definition 8.18. We let (A, k) be the locally constant stratified factorization algebra on I^n defined by the augmentation $\epsilon : \mathcal{A} \to k$ that is $(A, k) = \Upsilon(\epsilon)$.

Since Υ is a functor, then (A, k) is functorial with respect to maps of augmented E_n-algebras. More precisely, we have the faithful functor

(108) $$\tilde{\Upsilon} : E_n\text{-Alg}^{aug} \longrightarrow Hom_{E_n\text{-Alg}} \longrightarrow \text{Fac}_{I^n}^{lc}.$$

Remark 8.19. The factorization algebra (A, k) is explicitly described as follows. Let $\mathcal{U}_{I^n}$ be the (factorizing) basis of opens consisting of all open subset $U \subset D^n \subset I^n$, and all open half-disk D; recall that we call a half-disk of I^n an open $D \subset I^n$ such that is there is an homeomorphism $\theta : D \overset{\cong}{\to} \tilde{D} \times [0,1)$ with $D \cap \partial I^n = \theta^{-1}(\tilde{D} \times \{0\})$.

Lemma 8.20. *The factorization algebra (A, k) of Definition 8.18 satisfies that:*
  (1) *for any $U \subset D^n \subset I^n$, one has $(A, k)(U) = \mathcal{A}(U)$;*
  (2) *for any half-disk $D \in \mathcal{U}_{I^n}$, one has $(A, k)(D) = k(D) = k$.*

---

[47]the reader shall be aware that the notation D^n in [G2] is what we denote I^n in the present paper

(3) *The restriction to $\mathcal{U}_{I^n}$ of the structure maps $\rho_{U_1,\ldots,U_r,V} : \bigotimes_{i=1}^{r}(A,k)(U_i) \rightarrow (A,k)(V)$ is given as follows: if $U_1,\ldots,U_r \in \mathcal{U}_{I^n}$ are pairwise disjoint open lying in $V \in \mathcal{U}_{I^n}$, then*
- *if $V \subset D^n$, one has*

$$\rho_{U_1,\ldots,U_r,V} = \bigotimes_{i=1}^{r}(A,k)(U_i) \cong \bigotimes_{i=1}^{r}\mathcal{A}(U_i) \overset{\rho^{\mathcal{A}}_{U_1,\ldots,U_r,V}}{\longrightarrow} (A,k)(V)$$

where the last map is the structure map of the factorization algebra $\mathcal{A}$;
- *if V is a half-disk, $U_1,\ldots,U_i \in D^n$ $(0 \le i \le r)$ and $U_{i+1},\ldots,U_r$ are half-disks, one has*

$$\rho_{U_1,\ldots,U_r,V} = \bigotimes_{j=1}^{r}(A,k)(U_j) \cong \Big(\bigotimes_{j=1}^{i}\mathcal{A}(U_j) \Big) \otimes \Big(\bigotimes_{j=i+1}^{r} k(U_j) \Big)$$

$$\overset{(\otimes_{j=1}^{i}\epsilon)\otimes id}{\longrightarrow} \bigotimes_{j=1}^{r} k(U_i) \overset{\rho^{k}_{U_1,\ldots,U_r,V}}{\longrightarrow} k$$

where the last map is the structure maps of the factorization algebra associated to k (Example 2.21).

Further, any $\mathcal{U}_{I^n}$-prefactorization algebra (in particular any factorization algebra on on I^n) satisfying (1), (2) and (3) above is a $\mathcal{U}_{I^n}$-factorization algebra and equivalent to (A,k).

Proof. Since $\mathcal{U}_{I^n}$ is a factorizing, stable by finite intersection, basis of opens, the uniqueness in the last claim follows from Proposition 2.28. Since (A,k) is a factorization algebra, then it is a $\mathcal{U}_{I^n}$-factorization algebra and the last claim now follows from the first one.

The restriction to $D^n = I^n \setminus \partial I^n$ of (A,k) is just the factorization algebra $\mathcal{A}$ by [G2, Proposition 30]. Now, by [G2, Example 37 6], the structure maps when V is a half-disk is given by the left module structure of $\int_{(\partial I^n)\times(-\infty,0]} k$ over $int_{(\partial I^n)\times(-\infty,0]}A$. The formula for the structure maps now follows from [G2, Corollary 6] and the factorization algebra formula for left modules as in [G2, § 6.1]. $\square$

The ground ring k is trivially augmented and the augmentation $\epsilon : A \rightarrow k$ is a map of augmented algebras. Applying the functor $\tilde{\Upsilon}$ defined by the composition (108) above, we get the following map of factorization algebras:

$$(109) \qquad\qquad \tilde{\Upsilon}(\epsilon) : (A,k) \longrightarrow (k,k).$$

An immediate consequence of Lemma 8.22 and Example 2.21 is that $(k,k) = k$. We thus obtain

Lemma 8.21. *Let $\epsilon : A \rightarrow k$ be an augmented E_n-algebra.*

- *The factorization algebra (A,k) has a canonical augmentation $(A,k) \overset{\tilde{\Upsilon}(\epsilon):}{\longrightarrow} (k,k) = k$.*
- *If $D \in \mathcal{U}_{I^n}$, then $\underline{\epsilon}(D) : (A,k)(D) \rightarrow k(D) = k$ is equal to $\epsilon(D)(D) : \mathcal{A}(D) \rightarrow k$ if $D \subset D^n$ and to the identity otherwise.*

Proof. The first claim was established right before the lemma. The second claim is a direct derivation of the construction of the functor (107): $\Upsilon : Hom_{E_n\text{-Alg}} \longrightarrow \text{Fac}_{I^n}^{lc}$ in [G2] and Lemma 8.22. $\square$

Assume now that $\epsilon : A \to k$ is an augmented E_m-algebra, with $m \geq n$ and let again $\epsilon : \mathcal{A} \to k$ be a map of locally constant factorization algebras representing it. By Theorem 2.29, then $\epsilon : \mathcal{A} \to k$ can be seen as a map in $E_n\text{-Alg}(\text{Fac}_{\mathbb{R}^{m-n}}^{lc}) \cong E_n\text{-Alg}(E_{m-n}\text{-Alg})$. From Definition 8.18, the factorization algebra (A, k) then belongs to $\text{Fac}_{I^n}^{lc}(E_{m-n}\text{-Alg})$.

Lemma 8.22. *Let $\epsilon : A \to k$ be an augmented E_m-algebra and $(A, k) \in \text{Fac}_{I^n}^{lc}(E_{m-n}\text{-Alg})$ be the associated factorization algebra (Definition 8.18). The factorization homology of (A, k) is equivalent (naturally with respect to maps of augmented E_m-algebras) in $E_{m-n}\text{-Alg}$ to the iterated bar construction of A:*

$$p_*(A, k) \xleftarrow{\cong} \int_{I^n \times \mathbb{R}^{m-n}} (A, k) \cong Bar^{(n)}(A).$$

Proof. A similar result can be found in [F1, AFT]. Let $q : I^n \to [0, 1]$ be the supremum norm map: $(x_1, \ldots, x_n) \mapsto \max(|x_i|)$. We thus have the factorization algebra $q_*(A, k) \in \text{Fac}_{[0,1]}(E_{m-n}\text{-Alg}) \cong \text{Fac}_{[0,1]}(\text{Fac}_{\mathbb{R}^{m-n}}^{lc})$. By [G2, § 6.1], $q_*(A, k)$ is a stratified locally constant. Here, we see $[0, 1]$ as being stratified with two 0-dimensional strata given by the points $\{0\}$ and $\{1\}$. The algebra corresponding to the open dimension 1 stratum (as in [G2, Proposition 26]) is $\int_{(\partial I^n) \times (-1,1) \times \mathbb{R}^{m-n}} A$, while the right module corresponding to the stratum $\{0\}$ is $A \cong \int_{D^n \times \mathbb{R}^{m-n}}$ and the left module corresponding to $\{1\}$ is $k = \int_{(\partial I^n) \times \mathbb{R}^{m-n}} k$ (by Example 2.21).

By definition, the factorization homology of (A, k) is the same as the factorization homology of $q_*(A, k) \in \text{Fac}_{[0,1]}^{lc}(E_{m-n}\text{-Alg})$:

$$\begin{aligned}
p_*(A, k) \cong \int_{I^n \times \mathbb{R}^{m-n}} (A, k) \quad &\cong \quad \int_{[0,1]} q_*(A, k) \\
&\cong \quad \int_{D^m \times \mathbb{R}^{m-n}} A \overset{\mathbb{L}}{\underset{\int_{S^{m-1} \times \mathbb{R}^{m-n+1}} A}{\otimes}} k
\end{aligned}$$

where the last line comes from [G2, Proposition 26]. All the equivalences are further natural with respect to augmented E_m-algebras maps since p_*, q_* are functors and by *loc. cit.*. The result now follows from Lemma 8.17. $\square$

We now derive another factorization algebra model for the Bar construction. Let $\widehat{D^n} = S^n$ be the one point compactification of D^n and let $\kappa : I^n \to S^n = \widehat{D^n}$ be the canonical projection collapsing the boundary ∂I^n to a point. We endow $\widehat{D^n} = S^n$ with the stratification with one dimension 0 stratum given by the point at infinity and one dimension n stratum. This way, $\kappa : I^n \to S^n = \widehat{D^n}$ is a map of stratified spaces (that is maps strata onto strata).

Definition 8.23. We let $\widehat{\mathcal{A}} \in \text{Fac}_{S^n}$ be the factorization algebra $\kappa_*((A, k))$ obtained by pushforward along $\kappa : I^n \to S^n = \widehat{D^n}$ of the factorization algebra (A, k) of Definition 8.18.

The pushforward $\kappa_*(k)$ of the trivial factorization algebra k on I^n is equal to the trivial factorization algebra k on $\widehat{D^n}$. Hence the pushforward along κ of the augmentation of (A, k) (Lemma 8.21), that is the map

$$(110) \qquad\qquad \widehat{\epsilon} : \widehat{A} = \kappa_*((A, k)) \xrightarrow{\kappa_*\left(\tilde{\Upsilon}(\epsilon)\right)} k$$

is an augmentation for $\widehat{\mathcal{A}}$.

Assume now that $\epsilon : A \to k$ is an augmented E_m-algebra, with $m \geq n$ and let again $\epsilon : \mathcal{A} \to k$ be a map of locally constant factorization algebras representing it. We then get that the factorization algebra $\widehat{\mathcal{A}}$ belongs[48] to $\mathrm{Fac}_{S^n}(\mathrm{Fac}_{\mathbb{R}^{m-n}})$.

Lemma 8.24. *Let $\epsilon : A \to k$ be an augmented E_m-algebra $(m \geq n)$ represented by a map $\epsilon : \mathcal{A} \to k$ of factorization algebras and $\widehat{\mathcal{A}}$ be given by Definition 8.23.*

The factorization homology of $\widehat{\mathcal{A}}$ is equivalent in E_{m-n}-Alg to the iterated bar construction of A:

$$p_*(\widehat{\mathcal{A}}) \cong \int_{I^n \times \mathbb{R}^{m-n}} (A, k) \xleftarrow{\simeq} Bar^{(n)}(A).$$

This equivalence is natural with respect to maps of augmented E_m-algebras.

Proof. Since p_*, κ_* are functors, by Lemma 8.22 we have an natural equivalence

$$p_*(\widehat{\mathcal{A}}) = p_*(\kappa_*((A, k))) = p_*((A, k)) \xleftarrow{\simeq} \int_{I^n \times \mathbb{R}^{m-n}} (A, k) \cong Bar^{(n)}(A).$$

$\square$

We now describe in more details $\widehat{\mathcal{A}}$. Recall that we see $S^n = \widehat{D^n} = D^n \cup \{\infty\}$ as a stratified space with one stratum being $D^n \subset \widehat{D^n}$ and the other one being its point at infinity. A basis of neighborhood of ∞ is given by the complements of closed *Euclidean* disk centered at 0 in D^n. In particular, we have a factorizing and stable by finite intersection basis $\mathcal{U}_{\widehat{D^n}}$ consisting of all opens $U \subset D^n \subset \widehat{D^n}$ and all opens which are the complement $\widehat{D^n} \setminus \overline{D}$ of a non-empty Euclidean disk $\overline{D} \subset D^n$ whose center is 0.

Proposition 8.25. *The factorization algebra $\widehat{\mathcal{A}}$ of Definition 8.23 satisfies that:*

(1) *for any $U \subset D^n \subset \widehat{D^n} = S^n$, one has $\widehat{\mathcal{A}}(U) = \mathcal{A}(U)$;*

(2) *for any compact disk[49] $\overline{D} \subset D^n$, one has as an natural equivalence*

$$\widehat{\mathcal{A}}(\widehat{D^n} \setminus \overline{D}) \cong k(\widehat{D^n} \setminus \overline{D}) = k.$$

(3) *The restriction to $\mathcal{U}_{\widehat{D^n}}$ of the structure maps $\rho_{U_1,\ldots,U_r,V} : \bigotimes_{i=1}^{r} \widehat{\mathcal{A}}(U_i) \to \widehat{\mathcal{A}}(V)$ is given as follows: if $U_1,\ldots,U_r \in \mathcal{U}_{\widehat{D^n}}$ are pairwise disjoint open lying in $V \in \mathcal{U}_{\widehat{D^n}}$, then*
  - *if $V \subset D^n$, one has*

$$\rho_{U_1,\ldots,U_r,V} = \bigotimes_{i=1}^{r} \widehat{\mathcal{A}}(U_i) \cong \bigotimes_{i=1}^{r} \mathcal{A}(U_i) \xrightarrow{\rho^{\mathcal{A}}_{U_1,\ldots,U_r,V}} \widehat{\mathcal{A}}(V)$$

  the last map being the structure map of the factorization algebra $\mathcal{A}$;

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[48] by abuse of notation we keep the notation $\widehat{\mathcal{A}}$ for the factorization algebra $\pi_*(\widehat{\mathcal{A}}) \in \mathrm{Fac}_{S^n}(\mathrm{Fac}_{\mathbb{R}^{m-n}})$

[49] by a compact disk of D^n, we mean the image in D^n of an embedding of the closed unit Euclidean disk

- *if $V = \widehat{D^n} \setminus \overline{D}$, where D is an Euclidean disk centered at 0, $U_1, \ldots, U_i \in D^n$ $(0 \le i \le r)$ and $U_{i+1}, \ldots, U_r$ are complements of Euclidean disks centered at 0[50], one has*

$$\rho_{U_1,\ldots,U_r,V} = \bigotimes_{j=1}^{r} \widehat{\mathcal{A}}(U_j) \cong \Big(\bigotimes_{j=1}^{i} \mathcal{A}(U_j) \Big) \otimes \Big(\bigotimes_{j=i+1}^{r} k(U_j) \Big)$$

$$\xrightarrow{(\otimes_{j=1}^{i} \epsilon) \otimes id} \bigotimes_{j=1}^{r} k(U_i) \xrightarrow{\rho^{k}_{U_1,\ldots,U_r,V}} k$$

   where the last map is the structure map of the factorization algebra associated to k.

(4) $\widehat{\mathcal{A}}$ *is stratified locally constant on $\widehat{D^n} = S^n$ (stratified by $\{\infty\} \subset D^n \cup \{\infty\} = S^n$ as above).*

(5) *Any $\mathcal{U}_{\widehat{D^n}}$-prefactorization algebra satisfying (1), (2) and (3) above is a $\mathcal{U}_{\widehat{D^n}}$-factorization algebra whose unique[51] extension as a factorization algebra is further equivalent to $\widehat{\mathcal{A}}$.*

Point **(5)** implies that $\widehat{\mathcal{A}}$ is the unique factorization algebra on $\widehat{D^n}$ satisfying the properties **(1)**, **(2)** and **(3)** of Proposition 8.25.

Remark 8.26. Note also that by Point.**(3)**, the equivalence, for any compact sub-disk $\overline{D}$, $\widehat{\mathcal{A}}(\widehat{D^n} \setminus \overline{D}) \cong k$ of Point.**(2)**, is induced by the augmentation map (110). Namely, let $\mathcal{V}$ be a factorizing cover of $\widehat{D^n} \setminus \overline{D}$. For any open subset $V \in \mathcal{V}$, we have the augmentation $\widehat{\epsilon} : \widehat{\mathcal{A}}(V) \to k(V) = k$ which is a map of factorization algebras, hence induces a map of Čech complexex: $\check{C}(\widehat{\mathcal{A}}, \mathcal{V}) \to \check{C}(k, \mathcal{V})$. The following diagram, in which the lower arrow is the equivalence of Proposition 8.25.**(2)**,

(111)
$$\begin{array}{ccc} \check{C}(\widehat{\mathcal{A}}, \mathcal{V}) & \xrightarrow{\;\widehat{\epsilon}\;} & \check{C}(k, \mathcal{V}) \\ \downarrow & & \downarrow \\ \widehat{\mathcal{A}}(\widehat{D^n} \setminus \overline{D}) & \xrightarrow{\;\simeq\;} & k \end{array}$$

is commutative in $k\text{-Mod}_\infty$ (as proved in the proof of Proposition 8.25). In particular, the lower map in the diagram is just $\widehat{\epsilon}(\widehat{D^n} \setminus \overline{D}) : \widehat{\mathcal{A}}(\widehat{D^n} \setminus \overline{D}) \longrightarrow k(\widehat{D^n} \setminus \overline{D}) = k$.

Proof of Proposition 8.25. Since $\widehat{\mathcal{A}}(U) = (A, k)(\kappa^{(-1)}(U))$, point **(1)** is immediate from Lemma 8.22.

By the generalized Schoenflies Theorem, the complement $\widehat{D^n} \setminus \overline{D}$ is homeomorphic to a disk. Consequently, claim **(2)** boils down to proving that $(A, k)((\partial I^n) \times [0, r)) \cong k$ (that is, we are left to the case where $\overline{D}$ is a compact Euclidean disk). The open set $(\partial I^n) \times [0, r)$ has a factorizing cover, stable by finite intersection, which consists of open half-disks of the form $C \times [0, r)$ (where, for instance C is a small convex sub-disk of ∂I^n); denote $\mathcal{H}$ such a cover.

Since (A, k) is a factorization algebra, we have that $(A, k)((\partial I^n) \times [0, r))$ is computed by the Čech complex $\check{C}(\mathcal{H}, (A, k))$ of this cover consisting of open half-disks. By Lemma 8.20.**(2)**, the value of (A, k) on any half-disk is just k; further the

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[50]note that the U_i's being disjoint implies $i = r$ or $r - 1$

[51]by Proposition 2.28

structure maps are those of the trivial factorization algebra k (see Example 2.21). Thus the Čech complex $\check{C}\big(\mathcal{H}, (A, k)\big)$ is the same as the one of the factorization algebra k, that is, $\check{C}\big(\mathcal{H}, (A, k)\big) = \check{C}\big(\mathcal{H}, k\big)$. Now, Lemma 2.22 implies that

$$(A, k)\big((\partial I^n) \times [0, r)\big) \cong k\big((\partial I^n) \times [0, r)\big) \cong CH_{(\partial I^n) \times [0,r)}(k) \cong k,$$

and, combining this with the definition of the augmentation map (110) and Lemma 8.21, we further obtain a commutative diagram

$$
\begin{array}{ccc}
\check{C}\big(\mathcal{H}, (A, k)\big) & \xrightarrow{\;\;\cong\;\;} & \check{C}(k, \mathcal{H}) \\
\downarrow & & \downarrow \\
\widehat{\mathcal{A}}\big(\widehat{D^n} \setminus \overline{D}\big) & \xrightarrow{\;\;\widehat{\epsilon}\;\;} & k(\widehat{D^n} \setminus \overline{D}) = k
\end{array}
$$

which finishes to prove claim **(2)**. In particular we see that the natural equivalence in claim **(2)** is induced by the augmentation. This implies in particular the commutativity of diagram (111) above and thus Remark 8.26 as well.

Claim **(3)** is proved as in Lemma 8.20.

Let V be an open disk containing ∞. Then $V \setminus \{\infty\}$ is homeomorphic to $\mathbb{R}^n \setminus \{0\}$ and $\kappa^{-1}(V)$ is homeomorphic to $(\partial I^n) \times [0, 1)$. Thus statement **(4)** reduces to statement **(2)**.

To prove claim **(5)**, note that $\mathcal{U}_{\widehat{D^n}}$ is a factorizing, stable by finite intersection, basis of opens and $\widehat{A}$ is a prefactorization algebra satisfying claims **(1)**, **(2)** and **(3)**. Since we already know that $\widehat{A}$ is a factorization algebra, we know that the data given by the claims **(1)**, **(2)** and **(3)** does define a $\mathcal{U}_{\widehat{D^n}}$-factorization algebra. Proposition 2.28 implies that any factorization algebra whose value on $\mathcal{U}_{\widehat{D^n}}$ agrees with the one of $\widehat{A}$ (which is given by claims **(1)**, **(2)** and **(3)**) is equivalent to $\widehat{A}$ which terminates the proof of statement **(5)**. $\qquad\square$

8.2.3. *The E_n-coalgebra structure of the iterated Bar construction.* In this section we prove that the iterated Bar construction $Bar^{(n)}(A)$ of an augmented E_n-algebra has an E_n-coalgebra structure. In view of Theorem 2.29, it is equivalent to prove that there exists a locally constant factorization algebra on $\mathbb{R}^n$ whose global section is the iterated Bar construction $Bar^{(n)}(A)$. This is the approach we follow here.

Remark 8.27 (*sketch of the construction*)**.** The result of Lemma 8.24 and Proposition 8.25 is that the Bar construction of an augmented E_n-algebra $\epsilon : A \to k$ is the global section (i.e. factorization homology) of the stratified locally constant factorization algebra on the Alexandroff compactification $D^n \cup \{\infty\} = \widehat{D^n} = S^n$ of D^n whose value on D^n is just the one of A and whose value in a disk centered at ∞ is just k. For any disk D inside D^n, we can also form its Alexandroff one-point compactification $\widehat{D} = \{\infty\} \cup D$ and by restriction to D, the factorization algebra associated to A will give rise to a stratified locally constant factorization algebra on D^n. The procedure can be done simultaneously for pairwise disjoint opens in D^n; this suggest how the iterated bar construction gives rise to a factorization coalgebra on D^n. We now make this scheme precise.

Let $\epsilon : A \to k$ be a map of E_m-algebras (with $m \geq n$) and which we assume to be represented by a map $\epsilon : \mathcal{A} \to k$ of factorization algebras over $\mathbb{R}^m$. In other words, $\mathcal{A}(D) \cong \int_D A$ for any disk $D \subset \mathbb{R}^m$. Recall from Theorem 2.30 that $\pi_*(\mathcal{A}) \in \mathrm{Fac}^{lc}_{\mathbb{R}^n}(\mathrm{Fac}^{lc}_{\mathbb{R}^{m-n}})$. In the following, in order to shorten notations, we will simply write $\mathcal{A}$ for $\pi_*(\mathcal{A})$. In particular for any U open subset of $\mathbb{R}^n$, $\mathcal{A}(U \times \mathbb{R}^n) = \int_{U \times \mathbb{R}^{m-n}} A$ inherits an E_{m-n}-algebra structure (canonically induced by a locally constant factorization algebra structure on $\mathbb{R}^{m-n}$).

The *restriction of $\epsilon : \mathcal{A} \to k$ to $U \times \mathbb{R}^{m-n}$ is an augmentation for $\mathcal{A}(U \times \mathbb{R}^{m-n})$.*

Remark 8.28. Let $\phi : \mathbb{R}^n \overset{\sim}{\rightrightarrows} U \subset \mathbb{R}^n$ be an embedding of a disk in $\mathbb{R}^n$. Then we have an homeomorphism $\phi \times id : \mathbb{R}^n \times \mathbb{R}^{m-n} \overset{\sim}{\rightrightarrows} U \times \mathbb{R}^{m-n}$ which makes $\mathcal{A}_{|U \times \mathbb{R}^{m-n}}$ into a locally constant factorization algebra over $\mathbb{R}^m$ hence $\mathcal{A}(U \times \mathbb{R}^{m-n})$ is an E_m-algebra.

Definition 8.29. We denote $\mathcal{A}_\phi$ the *augmented E_m-algebra $\mathcal{A}(U \times \mathbb{R}^{m-n}) \cong \int_{U \times \mathbb{R}^m} A$.*

We can thus define $Bar^{(n)}(A_\phi)$ *the n-fold iterated Bar construction of A_ϕ and $\widehat{\mathcal{A}_\phi} \in Fac^{lc}_{\widehat{U}}(E_{m-n}\text{-}Alg)$ the stratified locally constant factorization algebra of Definition 8.23.*

We wish to define a factorization algebra $\widehat{\mathcal{A}_U}$ on $\widehat{U} \cong U \cup \{\infty\}$ the Alexandroff compactification of U. We essentially proceed as for Definition 8.23 above:

Definition 8.30. Let $\mathcal{W}_{\widehat{U}}$ be the open cover[52] of $\widehat{U}$ consisting of all opens W such that either $W \subset U \subset \widehat{U}$ or else $W = \widehat{U} \setminus \overline{D}$ where $\overline{D} \subset U$ is any compact disk[53]

(1) for any $W \subset U \subset \widehat{U}$ and for any compact disk $\overline{D} \subset U$, set

$$\widehat{\mathcal{A}_U}(W) = \mathcal{A}(W), \qquad \widehat{\mathcal{A}_U}\big(\widehat{U} \setminus \overline{D}\big) = k\big(\widehat{U} \setminus \overline{D}\big) = k.$$

(2) Let $W_1, \ldots, W_r \subset \widehat{U}$ are pairwise disjoint opens lying in $V \in \widehat{U}$. Assume in addition that
   (a) either V is in $U \subset \widehat{U}$ (and thus so are all W_i);
   (b) or $V = \widehat{U} \setminus \overline{D}$, where D is a compact disk in U, $W_1, \ldots, W_i \in U$ and $W_{i+1}, \ldots, W_r$ are complements of compact disks (the W_i's being disjoint implies $i = r$ or $r - 1$).
   We define "structure maps" $\rho_{W_1, \ldots, W_r, V} : \bigotimes_{i=1}^r \widehat{\mathcal{A}_U}(W_i) \longrightarrow \widehat{\mathcal{A}_U}(V)$ as follows:
   • in case (2a), we set

$$\rho_{W_1, \ldots, W_r, V} = \bigotimes_{i=1}^r \widehat{\mathcal{A}}(W_i) \cong \bigotimes_{i=1}^r \mathcal{A}(W_i) \overset{\rho^{\mathcal{A}}_{W_1, \ldots, W_r, V}}{\longrightarrow} \widehat{\mathcal{A}}(V)$$

   the last map being the structure map of the factorization algebra $\mathcal{A}$;

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[52]which is not stable under intersection
[53]by a compact disk in U, we mean the image in U of an embedding of the closed unit Euclidean disk

- in case (2b), we set

$$\rho_{W_1,\dots,W_r,V} = \bigotimes_{j=1}^{r} \widehat{\mathcal{A}}(W_j) = \Big(\bigotimes_{j=1}^{i} \mathcal{A}(W_j) \Big) \otimes \Big(\bigotimes_{j=i+1}^{r} k(W_j) \Big)$$

$$\xrightarrow{(\otimes_{j=1}^{i} \epsilon) \otimes id} \bigotimes_{j=1}^{r} k(W_i) \xrightarrow{\rho_{W_1,\dots,W_r,V}^{k}} k$$

where the last map is the structure map of the factorization algebra associated to k.

Lemma 8.31.
- *There is a unique[54] factorization algebra $\widehat{\mathcal{A}_U}$ on $\widehat{U}$ which takes the values given by Definition 8.30.(1) and with structure maps specified by Definition 8.30. (2) above on the relevant opens.*
- *Further, $\widehat{\mathcal{A}_U}$ is stratified locally constant on $\widehat{U}$, which is stratified with one dimensions 0 stratum given by the point at ∞ and one dimension n stratum given by U.*

Proof. Since U is a disk, we can find an embedding $\phi : \mathbb{R}^n \xrightarrow{\sim} U \subset \mathbb{R}^n$, which induces an homeomorphism $\widehat{D^n} \cong \widehat{U}$. As in Definition 8.29, we have the E_m algebra $\mathcal{A}_\phi$ and a *stratified locally constant* factorization algebra on $\widehat{D^n} \cong \widehat{U}$. By Proposition 8.25, we see that the factorization algebra $\mathcal{A}_\phi$ takes the same value as $\widehat{A}_U$ on the opens specified in point (1). Further, it has the same structure maps as those given by Definition 8.30 (2) on the basis of opens $\Big(\phi(V), V \in \mathcal{U}_{\widehat{D^n}} \Big)$.

Thus by Proposition 8.25.(5), we see that $\mathcal{A}_\phi$ is the unique factorization algebra structure on $\widehat{U}$ taking these values.

It only remains to prove that $\widehat{\mathcal{A}_U}$ does has the structure maps claimed by Definition 8.30 (2) in case (2b) for arbitrary compact disk $\overline{D}$. The proof is similar to the proof of the commutativity of Diagram (111) obtained in the proof of Proposition 8.25; we used the generalized Schoenflies theorem to restrict to a cover by half disks and use the Čech complexes of this cover to deduce the result. $\square$

Definition 8.32. We denote $\widehat{\mathcal{A}_U}$ the stratified locally constant factorization algebra on $\widehat{U}$ defined by Lemma 8.31. It is augmented $\widehat{\epsilon} : \widehat{\mathcal{A}_U} \to k$.

Remark 8.33. Let $\phi : \mathbb{R}^m \xrightarrow{\sim} U$ be an homeomorphism so that we have the E_n-algebra $\mathcal{A}_\phi$ from Definition 8.29. By Lemma 8.24 and Lemma 8.31, we have an natural (with respect to maps of augmented E_m-algebras) equivalence of E_{m-n}-algebras:

$$(112) \qquad Bar^{(n)}(A_\phi) \xrightarrow{\simeq} p_*\big(\widehat{\mathcal{A}_\phi}\big) \cong \widehat{\mathcal{A}_U}(\widehat{U}).$$

The *one point compactification is contravariant* with respect to open inclusions: if $U \subset V$ are open subsets of $\mathbb{R}^n$, we have the continuous map

$$(113) \qquad \iota_U^V : \widehat{V} \longrightarrow \widehat{U}$$

which is the identity on $U \subset \widehat{V}$ and collapses the complement $\widehat{V} \setminus U$ to the point at ∞ of $\widehat{U} = U \cup \{\infty\}$.

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[54]up to a contractible family of choices

By pushing forward along ι_U^V, we get the factorization algebra

$$\iota_{U*}^V(\widehat{\mathcal{A}_V}) \in \mathrm{Fac}_{\widehat{U}}(\mathrm{Fac}_{\mathbb{R}^{m-n}}^{lc}).$$

Note that by definition of factorization homology for factorization algebras we have:

$$(114) \qquad \iota_{U*}^V(\widehat{\mathcal{A}_V})(\widehat{U}) \cong p_* \circ \iota_{U*}^V(\widehat{\mathcal{A}_V}) \cong p_*(\widehat{\mathcal{A}_V}) \cong \widehat{\mathcal{A}_V}(\widehat{V})$$

We wish to define a quasi-isomorphism $\gamma_U^V : \iota_{U*}^V(\widehat{\mathcal{A}_V}) \longrightarrow \widehat{\mathcal{A}_U}$ of factorization algebras over $\widehat{U}$.

To do this, we consider the cover of $\widehat{U}$ given consisting of all opens $W \subset U \subset \widehat{U}$ and all opens which are the complement $\widehat{U} \setminus \overline{D}$ of a compact disk $\overline{D}$. We note

Lemma 8.34. *Let D be an open subset of $\widehat{U}$.*

 (1) *If $D \subset U$, then $\iota_{U*}^V(\widehat{\mathcal{A}_V})(D) = \widehat{\mathcal{A}_V}(D) = \mathcal{A}(D) = \widehat{\mathcal{A}_V}(D)$;*
 (2) *if $D = \widehat{U} \setminus \overline{D}$, then $\iota_{U*}^V(\widehat{\mathcal{A}_V})(D) = \widehat{\mathcal{A}_V}(\widehat{V} \setminus \overline{D}) \cong k(\widehat{V} \setminus \overline{D}) = k$.*

Proof. Choose homeomorphisms $\phi : \mathbb{R}^n \xrightarrow{\sim} U$ and $\psi : \mathbb{R}^n \xrightarrow{\sim} V$ so that ψ identifies V with D^n and we are left to the case where $\phi : \mathbb{R}^n \to U$ is a sub-disk of D^n. We have a factorizing and stable by finite intersections basis $\mathcal{U}_{\widehat{U}}$ (as in Proposition 8.25). The basis $\mathcal{U}_{\widehat{U}}$ is defined as the set consisting of all opens $W \subset U \subset \widehat{U}$ and all opens which are the complement $\widehat{U} \setminus \phi(\overline{D})$ of the image by ϕ of a non-empty Euclidean compact disk $\overline{D} \subset \mathbb{R}^n$ whose center is 0.

Now, the lemma is a consequence of the Definition of the pushforward $\iota_{U*}^V(\widehat{\mathcal{A}_\psi})$, Definition 8.23 and Proposition 8.25.(1) and (2). $\qquad\square$

We now define the aforementioned map γ_U^V.

Lemma 8.35. *Let W be in the cover $\mathcal{W}_{\widehat{U}}$ as defined in Definition 8.30. (1); that is either $W \subset U \subset \widehat{U}$ or W is the complement $\widehat{U} \setminus \overline{D}$ of a compact disk[55] $\overline{D} \subset U$. Let*

$$\gamma_U^V(W) : \iota_{U*}^V(\widehat{\mathcal{A}_V})(W) \longrightarrow \widehat{\mathcal{A}_U}(W)$$

be the augmented E_{m-n}-algebra map defined (using the identifications provided by Lemma 8.34),

 • *as the identity map*

$$\gamma_U^V(W) : \iota_{U*}^V(\widehat{\mathcal{A}_V})(W) = \mathcal{A}(W) \xrightarrow{id} \mathcal{A}(W) = \widehat{\mathcal{A}_U}(W)$$

 if $W \subset U$;
 • *and, if $W = \widehat{U} \setminus \overline{D}$, $\overline{D} \subset U$ a compact disk, as the restriction of the augmentation of $\widehat{\mathcal{A}_V}$:*

$$\gamma_U^V(W) : \iota_{U*}^V(\widehat{\mathcal{A}_V})(W) = \widehat{\mathcal{A}_V}(\widehat{V} \setminus \overline{D}) \xrightarrow{\widehat{\epsilon}(\widehat{V} \setminus \overline{D})} k(\widehat{V} \setminus \overline{D}) = k = \widehat{A_U}(W)$$

 (where the last equality follows from Proposition 8.25 and $\widehat{\epsilon}$ is the augmentation map (110)).

 (1) *The collection $\left(\gamma_U^V(W)\right)_{W \in \mathcal{U}_{\widehat{U}}}$ is a map of $\mathcal{W}_{\widehat{U}}$-factorization algebras.*

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[55]by a compact disk in U, we mean the image in U of an embedding of the closed unit Euclidean disk

(2) *The collection* $\left(\gamma_U^V(W)\right)_{W\in\mathcal{U}_{\widehat{U}}}$ *has an* unique[56] *extension into* a map $\gamma_U^V :$ $\iota_{U*}^V(\widehat{\mathcal{A}_V}) \longrightarrow \widehat{\mathcal{A}_U}$ of factorization algebras over $\widehat{U}$.

(3) γ_U^V *is further a map of augmented factorization algebras (with respect to the augmentation* (110)*)*.

(4) *The map* $\iota_{U*}^V(\widehat{\mathcal{A}_V}) \xrightarrow{\gamma_U^V} \widehat{\mathcal{A}_U}$ *is an equivalence of factorization algebras.*

Proof. We need to prove that, for any pairwise disjoint open $W_1,\ldots,W_r \in \mathcal{W}_{\widehat{U}}$ lying in $Z \in \mathcal{W}_{\widehat{U}}$, the following diagram

(115)
$$\begin{array}{ccc} \bigotimes\limits_{i=1}^{r} \iota_{U*}^V(\widehat{\mathcal{A}_V})(W_i) & \xrightarrow{\rho_{W_1,\ldots,W_r,z}} & \iota_{U*}^V(\widehat{\mathcal{A}_V})(Z) \\ {\scriptstyle\bigotimes\limits_{i=1}^{r}\gamma_U^V(W_i)}\Big\downarrow & & \Big\downarrow{\scriptstyle\bigotimes\limits_{i=1}^{r}\gamma_U^V(Z)} \\ \bigotimes\limits_{i=1}^{r} \widehat{\mathcal{A}_U}(W_i) & \xrightarrow{\rho_{W_1,\ldots,W_r,z}} & \widehat{\mathcal{A}_U}(Z) \end{array}$$

is commutative.

If $Z \subset U$ (and consequently all the $W_i \subset U$ as well), then this is a trivial consequence of Lemma 8.34.(1). Let $Z = \widehat{U} \setminus \overline{K}$ with $\overline{K}$ a compact disk in U. We may assume $W_1,\ldots,W_j \in U$ with $j = r-1$ or $j = r$, and the remaining $W_{j+1},\ldots,W_r$ (note that there may be only one or zero such W_ℓ) to be of the form $\widehat{U}\setminus\overline{T}$ where $\overline{T}$ is a compact disk in U. Unfolding the definition of γ_U^V using statement (3) in Proposition 8.25 and Lemma 8.34, we obtain that the diagram (115) can be rewritten as the following diagram
(116)

$$\begin{array}{ccc} \big(\bigotimes\limits_{i=1}^{j}\widehat{\mathcal{A}_V}(W_i)\big)\otimes\big(\bigotimes\limits_{i=j+1}^{r}\widehat{\mathcal{A}_V}(\widehat{V}\setminus\overline{T})\big) & \xrightarrow{\rho_{W_1,\ldots,W_r,z}} & \widehat{\mathcal{A}_V}\big(\widehat{V}\setminus\overline{T}\big) \\ {\scriptstyle\big(\bigotimes\limits_{i=1}^{j}id\big)\otimes\big(\bigotimes\limits_{i=j+1}^{r}\widehat{\epsilon}(\widehat{V}\setminus\overline{T})\big)}\Big\downarrow & {\scriptstyle\bigotimes\limits_{i=1}^{r}\widehat{\epsilon}(W_i)}\searrow & \Big\downarrow{\scriptstyle\widehat{\epsilon}(\widehat{V}\setminus\overline{K})} \\ \big(\bigotimes\limits_{i=1}^{j}\widehat{\mathcal{A}_V}(W_i)\big)\otimes\big(\bigotimes\limits_{i=j+1}^{r}k\big) & \xrightarrow[\big(\bigotimes\limits_{i=1}^{j}\widehat{\epsilon}(W_i)\big)\otimes\big(\bigotimes\limits_{i=j+1}^{r}id\big)]{} & \bigotimes\limits_{i=1}^{r}k \xrightarrow{\rho_{W_1,\ldots,W_r,z}^k} k \end{array}$$

and that further the lower left triangle in diagram (116) is commutative. The commutativity of the upper right part of diagram (116) is given by the fact that $\widehat{\epsilon}\colon \widehat{\mathcal{A}_V} \to k$ is a map of factorization algebras. This proves that the $(\gamma_U^V(W))$ forms a map of $\mathcal{W}_{\widehat{U}}$-factorization algebras.

Now, note that $\mathcal{W}_{\widehat{U}}$ contains a factorizing, stable by finite intersections, basis $\mathcal{U}_{\widehat{U}}$ of opens. Indeed, let $\phi : \mathbb{R}^n \to U$ be an homeomorphism. Then the cover $\mathcal{U}_{\widehat{U}}$ consists of all opens $W \subset U$ and all opens $\widehat{U}\setminus\phi(K)$ where K is a compact Euclidean ball centered at 0 in $\mathbb{R}^n$. The fact that the collection $(\gamma_U^V(W))$ extends uniquely to a map of factorization algebras is hence a consequence of Proposition 2.28.

Indeed, if $\mathcal{F} \in \text{Fac}_{\widehat{U}}$ and $D \subset \widehat{U}$ is an open set, then the Čech complex $\check{C}(D_{\mathcal{U}_{\widehat{U}}}, \mathcal{F}) \cong \mathcal{F}(D)$ where $D_{\mathcal{U}_{\widehat{U}}}$ is the cover of D consisting of all opens of $\mathcal{U}_{\widehat{U}}$ which lies in D. In particular any map of $\mathcal{U}_{\widehat{U}}$-factorization algebras defines a map

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[56]up to a contractible family of choices

between the associated Čech complexes. This construction is the inverse of the restriction functors from factorization algebras to $\mathcal{U}_{\widehat{U}}$-factorization algebras.

From above, to prove statement.(3), it is sufficient to check that γ_U^V is a map of augmented factorization algebras on the opens of $\mathcal{U}_{\widehat{U}}$. If $W \subset U$, then there is nothing to prove since $\gamma_U^V(W)$ is the identity. If $W = \widehat{U} \setminus \phi(\overline{K})$ then $k = \widehat{\mathcal{A}_\phi}(W)$ and the augmentation map (110) is the identity $k \to k$ and there is nothing left to prove.

Finally, again by Proposition 2.28, to prove that $\iota_{U*}^V(\widehat{\mathcal{A}_\psi}) \xrightarrow{\gamma_U^V} \widehat{\mathcal{A}_\phi}$ is an equivalence of factorization algebras, it is sufficient to prove that its restriction $\gamma_U^V(W)$ on any open $W \in \mathcal{U}_{\widehat{U}}$ of the above basis is a quasi-isomorphism. The only case which needs a proof is when $W = \widehat{U} \setminus \overline{D}$ with $\overline{D} = \phi(K)$ where K is a compact Euclidean ball centered at 0 in $\mathbb{R}^n$. By Proposition 8.25.(2) and diagram (111), we have a commutative diagram

$$\iota_{U*}^V(\widehat{\mathcal{A}_V})(W) = \widehat{\mathcal{A}_V}(\widehat{V} \setminus \overline{D}) \xrightarrow{\gamma_U^V(W)} k(\widehat{V} \setminus \overline{D}) = k$$
$$\widehat{\epsilon}(\widehat{V} \setminus \overline{D}) \downarrow \cong \qquad \xrightarrow{id}$$
$$k$$

from which we deduce that $\gamma_U^V(W)$ is a quasi-isomorphism. Hence Claim.(4) of the Lemma holds. $\square$

Passing to factorization homology, *i.e.* evaluating on $\widehat{U}$, the factorization algebra map $\gamma_U^V : \iota_{U*}^V(\widehat{\mathcal{A}_V}) \longrightarrow \widehat{\mathcal{A}_U}$ induces a map $\gamma_U^V(\widehat{U}) : \iota_{U*}^V(\widehat{\mathcal{A}_V})(\widehat{U}) \longrightarrow \widehat{\mathcal{A}_U}(\widehat{U})$. Composing this map with the string of equivalences (114), we get the following map of (augmented) E_{m-n}-algebras

$$(117) \qquad \tilde{\gamma}_U^V : p_*(\widehat{\mathcal{A}_V}) \cong \widehat{\mathcal{A}_\psi}(\widehat{V}) \cong \iota_{U*}^V(\widehat{\mathcal{A}_V}) \xrightarrow{\gamma_U^V(\widehat{U})} \widehat{\mathcal{A}_U} \cong p_*(\widehat{\mathcal{A}_U})$$

between the factorization homology of $\widehat{\mathcal{A}_V}$ and the factorization homology of $\widehat{\mathcal{A}_U}$.

We now define the factorization *coalgebra* (Definition 2.12) $U \mapsto Bar^{(n)}(A)(U)$ we have been seeking for.

Definition 8.36. Let $\epsilon : \mathcal{A} \to k$ be a map of locally constant factorization algebras over $\mathbb{R}^m$.

- Let $U \subset \mathbb{R}^n$ be a disk. We define $Bar^{(n)}(\mathcal{A})(U) := p_*(\widehat{\mathcal{A}_U}) \in E_{m-n}\text{-Alg}^{aug}$ the factorization homology of the Factorization algebra $\mathcal{A}_U$ on $\widehat{U}$ from Definition 8.32.
- Let $U_1, \ldots, U_r$ be a family pairwise disjoint open sub-disks of an open disk $V \subset \mathbb{R}^n$. We define the structure map $\delta_{U_1,\ldots,U_r,V} : Bar^{(n)}(A)(V) \longrightarrow \bigotimes_{i=1}^r Bar^{(n)}(A)(U_i)$ to be the following maps in $E_{m-n}\text{-Alg}^{aug}$:

$$\delta_{U_1,\ldots,U_r,V} : Bar^{(n)}(A)(V) = p_*(\widehat{\mathcal{A}_V})$$

$$\xrightarrow{\overset{r}{\underset{i=1}{\bigotimes}} \tilde{\gamma}_{U_i}^V} \bigotimes_{i=1}^r p_*(\widehat{\mathcal{A}_{U_i}}) = Bar^{(n)}(A)(U_1) \otimes \cdots \otimes Bar^{(n)}(A)(U_r).$$

Here the maps $\tilde{\gamma}_{U_i}^V$ are the compositions (117).

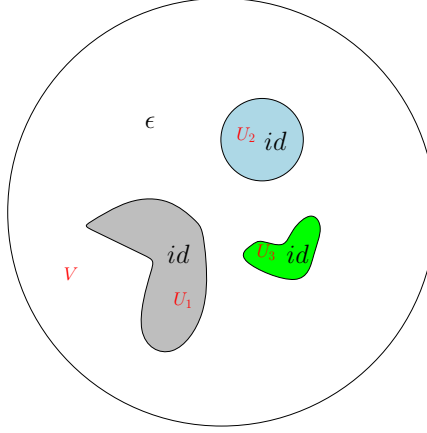FIGURE 3. The map $Bar^{(n)}(A)(V) \rightarrow Bar^{(n)}(A)(U_1) \otimes Bar^{(n)}(A)(U_2) \otimes Bar^{(n)}(A)(U_3)$

Let $\epsilon : A \to k$ be a map of E_m-algebras (with $m \geq n$) and assume it is represented[57] by the factorization algebra map $\epsilon : \mathcal{A} \to k$. In that case, *we also denote*

$$Bar^{(n)}(A)(U) := Bar^{(n)}(\mathcal{A})(U).$$

Unfolding the definition, the map $\delta_{U_1,\dots,U_r,V}$ is essentially the map of factorization algebra given by the identity on each U_i and the augmentation in their complement as is pictured in Figure 3 (in the case $r = 3$).

Theorem 8.37. *Let $0 \leq n \leq m$.*

(1) *There is an ∞-functor*

$$\mathcal{B}ar^{(n)} : Fac^{lc,aug}_{\mathbb{R}^n} \longrightarrow coFac^{lc}_{\mathbb{R}^n}\left(Fac^{lc,aug}_{\mathbb{R}^{m-n}} \right)$$

from $(\infty,1)$-category of locally constant augmented factorization algebras over $\mathbb{R}^m$ to the $(\infty,1)$-category of locally constant cofactorization algebras[58] over $\mathbb{R}^n$ with values in locally constant augmented factorization algebras over $\mathbb{R}^{m-n}$.

The functor $\mathcal{B}ar^{(n)}$ is given by the rule $\phi \mapsto Bar^{(n)}(A)(U)$ together with the structure maps $\delta_{U_1,\dots,U_r,V}$ of Definition 8.36.

(2) *let $\epsilon : A \to k$ be an augmented E_m-algebra. There is an natural equivalence $Bar^{(n)}(A) \xrightarrow{\simeq} Bar^{(n)}(A)(\mathbb{R}^n)$ between the iterated Bar construction of A (in the sense of Definition 8.15) and the cofactorization homology of $\mathcal{B}ar^{(n)}(\mathcal{A})$.*

In particular, the iterated Bar construction $Bar^{(n)}(A)$ has an natural structure of E_n-coalgebra in E_{m-n}-Alg^{aug} and the iterated Bar construction functor (Definition 8.15) lifts as a functor of $(\infty,1)$-categories

$$Bar^{(n)} : E_m\text{-}Alg^{aug} \longrightarrow E_n\text{-}coAlg\left(E_{m-n}\text{-}Alg^{aug} \right).$$

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[57]in other words $\mathcal{A}(W) \cong \int_W A$ for any open subset $W \subset \mathbb{R}^n$
[58]in the sense of Definition 2.12, that is a locally constant $N(\mathrm{Disk})(\mathbb{R}^n)$-coalgebra

Proof. First note that $Bar^{(n)}(A)(\mathbb{R}^n) = p_*(\widehat{\mathcal{A}})$ where $\widehat{\mathcal{A}}$ is the factorization algebra on $\mathbb{R}^n$ of Definition 8.23. Thus, by Lemma 8.24, we have an natural (with respect to maps of augmented E_m-algebras) equivalence

$$Bar^{(n)}(A) \xrightarrow{\simeq} p_*(\widehat{A}) = Bar^{(n)}(A)(\mathbb{R}^n).$$

Hence, part **(2)** in the Theorem is a corollary of part **(1)** and the relationship between factorization (co)-algebras an E_ℓ-(co)-algebras, namely Theorem 2.29 and Proposition 2.25.

We now prove part (1). The functoriality of $U \mapsto Bar^{(n)}(\mathcal{A})(U)$ is a straightforward consequence of the functoriality of $p_*(\widehat{\mathcal{A}_U})$ and of the transformations γ_U^V of Lemma 8.35.

Now, recall that each of the maps $\tilde{\gamma}_{U_i}^V$ (defined as the composition (117)) are augmented E_{m-n}-algebras maps. Hence so is the map

$$\delta_{U_1,\ldots,U_r,V} : Bar^{(n)}(\mathcal{A})(V) \longrightarrow Bar^{(n)}(\mathcal{A})(U_1) \otimes \cdots \otimes Bar^{(n)}(\mathcal{A})(U_r)$$

from Definition 8.36.

The invariance under the symmetric group action of the structure map follows right away from its definition. We also need to check the naturality of the structure maps with respect to inclusions of disks, *i.e.*, the identity:

$$(118) \quad \left(\delta_{W_1^1,\ldots,W_{i_1}^1,U_1} \otimes \cdots \otimes \delta_{W_1^r,\ldots,W_{i_r}^r,U_r} \right) \circ \delta_{U_1,\ldots,U_r,V} = \delta_{W_1^1,\ldots,W_{i_1}^1,\ldots,W_1^r,\ldots,W_{i_r}^r,V},$$

which has to hold for any families of pairwise disjoint open sub-disks $W_i^j \subset U_j$ (where $j = 1 \ldots r$). Unfolding Definition 8.36, we see that the identity (118) follows from the following identity

$$(119) \qquad \qquad \tilde{\gamma}_{W_i^j}^{U_j} \circ \iota_{U_j *}^V \left(\tilde{\gamma}_{U_j}^V \right) = \tilde{\gamma}_{W_i^j}^V$$

if it holds for all inclusions $W_i^j \subset U_j \subset V$ of open subsets. It is enough to check this identity for the underlying factorization algebras maps, that is too prove:

$$(120) \qquad \qquad \gamma_{W_i^j}^{U_j} \circ \iota_{U_j *}^V \left(\gamma_{U_j}^V \right) = \gamma_{W_i^j}^V.$$

Let $\theta_i^j : \mathbb{R}^n \to W_i^j$ and $\phi_j : \mathbb{R}^n \xrightarrow{\simeq} U_j \subset V$ be homeomorphisms. In view of Lemma 8.35, it is sufficient to check the above identity (120) on the factorizing cover $\mathcal{U}_{\widehat{W_i^j}}$ consisting of all opens in W_i^j and all complements of $\theta_i^j(\overline{D})$ where D is a compact Euclidean disk. Both sides of identity (120) are equal to the identity when restricted to an open subsets of W_i^j and to the (restriction of the) augmentation in the second case since $\gamma_{U_j}^V$ is a map of augmented algebras (Lemma 8.35).

It remains to prove the locally constant condition. That is we need to see that for an open sub-disk $U \hookrightarrow V$ of a disk V, the map (117)

$$\tilde{\gamma}_U^V : Bar^{(n)}(\mathcal{A})(V) = p_*(\widehat{\mathcal{A}_V}) \cong \widehat{\mathcal{A}_\psi}(\widehat{V}) \cong \iota_{U *}^V(\widehat{\mathcal{A}_V})$$

$$\xrightarrow{\gamma_U^V(\widehat{U})} \widehat{\mathcal{A}_U} \cong p_*(\widehat{\mathcal{A}_U}) = Bar^{(n)}(A)(U)$$

is a quasi-isomorphism. That $\gamma_U^V(\widehat{U})$ is a quasi-isomorphism is given by Lemma 8.35.(4). Hence, so is $\tilde{\gamma}_U^V$.

We have proved that the rule $\phi \mapsto Bar^{(n)}(\mathcal{A})(\phi)$ (see construction (8.36)) is a locally constant $N(\text{Disk}(\mathbb{R}^n))$-coalgebra object in E_{m-n}-algebras. Consequently, the iterated Bar construction given by Definition 8.15 is a functor from augmented

E_m-algebras to E_n-coAlg$\Big(E_{m-n}$-Alg$^{aug}\Big)$. By Proposition 8.14, this functor agrees (in the $(\infty, 1)$-category E_{m-n}-Alg^{aug}) with the one given in Section 8.1. The identification of the two E_n-coalgebras structure is done as in the proof of Proposition 6.15. □

Remark 8.38 (*sketch of a variant*). Let A be an E_n-algebra induced by a factorization algebra $\mathcal{A}$ on $\mathbb{R}^n$. The factorization algebra $\widehat{\mathcal{A}}$ on S^n (from Definition 8.23) is obtained by pushing forward the factorization algebra (A, k) on the stratified closed disk I^n from Definition 8.18. Further, for convex bounded open subsets of $\mathbb{R}^n$, we can think of the iterated Bar construction of A, restricted on V as a stratified factorization algebra $D \mapsto Bar^{(n)}(A)(D)$ on the closure $\overline{V}$ of V (which assigns the A-E_n-module k to balls in a neighborhood of the boundary $\overline{V} \setminus V$). In fact, for any disk V and any homeomorphism $\psi : \mathbb{R}^n \xrightarrow{\cong} V$, we can construct a factorization algebra (A_ψ, k) on the stratified closed disk I^n and the global section of this factorization algebra is quasi-isomorphic to $Bar^{(n)}(A)$. It is possible to define this way a locally constant *parametrized* factorization algebra on $\mathbb{R}^n$ (Definition 2.24) which is equivalent as the one we construct using $\widehat{\mathcal{A}}$ in Theorem 8.37.

The basic idea is that, given sub-disks $U_1, \ldots, U_r$ in V with homeomorphisms $\phi_i : \mathbb{R}^n \xrightarrow{\cong} U_i$ and an embedding $h : \coprod_{i=1}^r \mathbb{R}^n \to \mathbb{R}^n$ such that $\psi \circ h = \coprod_{i=1}^r \phi_i$, we can construct a locally constant stratified factorization algebra $\mathcal{F}$ on I^n which is stratified with one open strata given by the union of the disks $h(\bigcup_{i=1}^r \mathbb{R}^n)$ and one closed stratum given by their complement $I^n \setminus h(\bigcup_{i=1}^r \mathbb{R}^n)$. Then $\mathcal{F}$ is roughly defined as the rule which to each ball D inside $\phi_i^{-1}(U_i)$ associates $\int_{\phi_i(D)} A$, and which associates $\mathcal{F}(D) = k$ on the closed strata. The factorization algebra structure is given by the A-E_n-module structure of k. The map which is the identity on each disk D inside the preimage of a U_i and is the augmentation $\epsilon : A \to k$ on each disk in a small neighborhood of the closed strata defines a map of factorization algebra $(A_\psi, k) \to \mathcal{F}$, which on the global section is a map from $Bar^{(n)}(A_\psi) \to \bigotimes_{i=1}^r Bar^{(n)}(A_{\phi_i})$.

Let $\epsilon : \mathcal{A} \to k$ be a map of augmented locally constant factorization algebras over $\mathbb{R}^m$ and $1 \le i, j$ be such that $i + j \le m$. By Theorem 8.37, we have the ith Bar construction $\mathcal{B}ar^{(i)}(\mathcal{A}) \in$ coFac$_{\mathbb{R}^i}^{lc,aug}\Big(Fac_{\mathbb{R}^{m-i}}^{lc,aug}\Big)$. In particular for every open set $U \in \mathbb{R}^i$, we get an augmented factorization algebra $\mathcal{B}ar^{(i)}(\mathcal{A})(U) \in$ Fac$_{\mathbb{R}^{m-i}}^{lc,aug}$ from which, by Theorem 8.37 again, we get

$$\mathcal{B}ar^{(j)}\big(\mathcal{B}ar^{(i)}(\mathcal{A})(U)\big) \in \text{coFac}_{\mathbb{R}^j}^{lc}\Big(\text{Fac}_{\mathbb{R}^{m-i-j}}^{lc,aug}\Big).$$

Recall that the structure maps $\delta_{U_1 \ldots, U_r, V}$ from Definition 8.36 (associated to the functor $\mathcal{B}ar^{(i)}$ and sub-disks U_i, V) are as the tensor product $\bigotimes_{i=1}^r \tilde{\gamma}_{U_i}^V$ where the $\tilde{\gamma}_{U_i}^V$ are maps of augmented factorization algebras over $\mathbb{R}^{m-i}$. Hence we get a map

$$(121) \quad \bigotimes_{i=1}^r \mathcal{B}ar^{(j)}\big(\tilde{\gamma}_{U_i}^V\big) : \mathcal{B}ar^{(j)}\big(\mathcal{B}ar^{(i)}(\mathcal{A})(V)\big)$$

$$\longrightarrow \mathcal{B}ar^{(j)}\big(\mathcal{B}ar^{(i)}(\mathcal{A})(U_1)\big) \otimes \cdots \otimes \mathcal{B}ar^{(j)}\big(\mathcal{B}ar^{(i)}(\mathcal{A})(U_r)\big)$$

in $\mathrm{coFac}^{lc}_{\mathbb{R}^j}\left(\mathrm{Fac}^{lc,aug}_{\mathbb{R}^{m-i-j}}\right)$. The proof of Theorem 8.37 and the proof of Lemma 8.17 shows that

Proposition 8.39. *Let $\epsilon : \mathcal{A} \to k$ be a map of augmented locally constant factorization algebras over $\mathbb{R}^m$ and $1 \le i, j$ be such that $i + j \le m$.*

(1) *The structure maps (121) make $\mathcal{B}ar^{(j)}\big(\mathcal{B}ar^{(i)}(\mathcal{A})\big)$ an object of the $(\infty,1)$-category $\mathrm{coFac}^{lc}_{\mathbb{R}^i}\left(\mathrm{coFac}^{lc}_{\mathbb{R}^j}\left(\mathrm{Fac}^{lc,aug}_{\mathbb{R}^{m-i-j}}\right)\right)$, functorially in $\mathcal{A}$: in other words we have a functor*

$$\mathcal{B}ar^{(j)} \circ \mathcal{B}ar^{(i)} : Fac^{lc,aug}_{\mathbb{R}^m} \longrightarrow coFac^{lc}_{\mathbb{R}^i}\left(coFac^{lc}_{\mathbb{R}^j}\left(Fac^{lc,aug}_{\mathbb{R}^{m-i-j}}\right)\right).$$

(2) *There is a commutative diagram of functors:*

$$
\begin{array}{ccc}
Fac^{lc,aug}_{\mathbb{R}^m} & \xrightarrow{\ \mathcal{B}ar^{(i+j)}\ } & coFac^{lc}_{\mathbb{R}^{i+j}}\left(Fac^{lc,aug}_{\mathbb{R}^{m-i-j}}\right) \\
 & \searrow{\scriptstyle \mathcal{B}ar^{(j)}\circ \mathcal{B}ar^{(i)}} & \downarrow{\scriptstyle \simeq}\ \pi_* \\
 & & coFac^{lc}_{\mathbb{R}^i}\left(coFac^{lc}_{\mathbb{R}^j}\left(Fac^{lc,aug}_{\mathbb{R}^{m-i-j}}\right)\right)
\end{array}
$$

where the left vertical arrow is the pushforward.

In other words, through Dunn isomorphism, the proposition states that the functor $\mathcal{B}ar^{(n)}$ is the same as the n-times iterated Bar construction $\mathcal{B}ar^{(1)} \circ \cdots \circ \mathcal{B}ar^{(1)}$.

We finish this section by comparing the iterated Bar construction of Theorem 8.37 with centralizers and the construction of § 8.1.

Proposition 8.40. *Let $\epsilon : A \to k$ be an augmented E_m-algebra and $0 \le n \le m$.*

(1) *The dual $RHom(Bar^{(m)}(A), k)$, endowed with the E_m-algebra structure dual to the E_m-coalgebra structure of $Bar^{(m)}(A)$ (given by Theorem 8.37.(2)), is the centralizer $\mathfrak{z}(A \xrightarrow{\epsilon} k)$ of the augmentation (see § 6.3).*

(2) *$Bar^{(1)}(A)$ is equivalent as an E_1-coalgebra to the standard (§ 8.1) Bar construction $Bar^{std}(A)$ and $Bar^{(n)}(A)$ is equivalent to the iterated Bar constructions of [F1] (in the ∞-category $E_n\text{-}coAlg\big(E_{m-n}\text{-}Alg^{aug}\big)$).*

(3) *If $m = \infty$, the iterated Bar functor*

$$Bar^{(n)} : E_\infty\text{-}Alg^{aug} \longrightarrow E_n\text{-}coAlg\big(E_\infty\text{-}Alg^{aug}\big)$$

given by Theorem 8.37 is naturally equivalent to the one obtained in § 8.1 (and in particular Theorem 8.9).

Proof. Dualizing the construction of the locally constant $N(\mathrm{Disk}(\mathbb{R}^n))$-coalgebra structure shows that the dual $RHom(Bar^{(n)}(A), k)$ of the Bar construction has a locally constant $N(\mathrm{Disk}(\mathbb{R}^n))$-algebra structure whose global section gives us the E_m-algebra structure on $RHom(Bar^{(m)}(A), k)$ asserted in Claim (1)

By Proposition 2.28, it is enough to check that this dual structure coincides with the one given in Theorem 6.8 on the factorizing basis $\mathcal{CVX}$ of bounded convex open subsets of $\mathbb{R}^n$. Recall $\int_V k \cong k(V) = k$ for any open V. For $U \in \mathcal{CVX}$ with center

$*_U$, we have have an natural equivalence
(122)
$$Bar^{(m)}(A)(U) \;\cong\; p_*(\widehat{\mathcal{A}_U}) \;\cong\; \int_U A \underset{\int_{U\setminus\{*_U\}}A}{\overset{\mathbb{L}}{\otimes}} \int_{\widehat{U}\setminus\{*_U\}} k \;\cong\; \int_U A \underset{\int_{U\setminus\{*_U\}}A}{\overset{\mathbb{L}}{\otimes}} k$$

given by Lemma 8.24, Proposition 8.25 and Lemma 8.22 (this also follows from Remark 8.33 applied to any $\phi : \mathbb{R}^n \xrightarrow{\simeq} U$ such that $\phi(0) = *_U$). It follows that

$$(123) \qquad RHom(Bar^{(m)}(A)(U), k(U)) \;\cong\; RHom^{left}_{\int_{U\setminus\{*_U\}}A}\Big(\int_U A, \int_U k\Big)$$
$$\cong\; RHom^{\mathcal{E}_n}_A\big(A, k\big)(U)$$

where the last line is from Step 2, § 6.2.2 and the A-E_n-module structure on k is given by the augmentation $\epsilon : A \to k$. To conclude that the dual of $U \mapsto Bar^{(m)}(A)(U)$ is the factorization algebra of Theorem 6.8, it remains to the compare the dual of the structure maps of Definition 8.36 with the ones in § 6.2.2.

Let $U_1, \ldots, U_r$ be convex open sets lying inside a bounded convex open set V. By Lemma 8.35, the dual $RHom(\gamma^V_{U_i}, k)$ is a factorization algebra map on U_i which is given by the augmentation $\widehat{\epsilon}$ on every open subset $W = \widehat{U} \setminus \overline{D}$ which is the complement of a compact Euclidean disks containing the $*_i$. Further, on any open subset W' of such a W, the image under the equivalence (123) of $RHom(\gamma^V_{U_i}, k)$ evaluated on W', is a section in $\mathrm{Map}_{\mathrm{Fac}^{lc}_{U_i*_i}}(\mathcal{A}_{|U_i}, \mathcal{B}_{|U_i})$ (see § 6.2.2) which, again, is simply given by the augmentation.

A contrario, on any open set W lying inside U_i, the dual $RHom(\gamma^V_{U_i}, k)$ is the identity. Thus, its image under the equivalence (123) on any open subset $*_i \subset W \subset U_i$, is just the map taking a global section $f \in RHom^{\mathcal{E}_n}_A\big(A, k\big)(U_i)$ to its restriction on W.

Since $\delta_{U_1,\ldots,U_r}$ is obtained by tensor product of the $\gamma^V_{U_i}$ (Definition 8.36), it follows that the dual of $\delta_{U_1,\ldots,U_r}$ coincides with the structure maps $\rho_{U_1,d\ldots,U_r,V}$ given by Formula (37) on the cover $\mathcal{U}_{U_1,\ldots,U_r,V}$. This proves Claim (1).

That the algebraic of $Bar^{(n)}(A)$ agrees with the one in [F1] follows from Dunn Theorem (see [L-HA, F1] or Theorem 2.30) once we know that $Bar^{(1)}(A)$ is equivalent, as an E_1-coalgebra, to the standard Bar construction $Bar^{std}(A)$. By homotopy invariance, we may assume that A is a differential graded associative algebra. By Lemma 8.13, we have a natural equivalence $Bar(A) \cong Bar^{std}(A)$ and further the (two constructions) of the Bar construction computes the derived functor $k \otimes^{\mathbb{L}}_A k$. The coalgebra structure of $Bar^{std}(A)$ is induced by the comultiplication $\delta : Bar^{std}(A) \to Bar^{std}(A) \otimes Bar^{std}(A)$ which realized the following map of derived functors (in k-Mod_∞):

$$(124) \qquad \delta : k \otimes^{\mathbb{L}}_A k \cong k \otimes^{\mathbb{L}}_A A \otimes^{\mathbb{L}}_A k \xrightarrow{id\otimes^{\mathbb{L}}_A \epsilon \otimes^{\mathbb{L}}_A id} k \otimes^{\mathbb{L}}_A k \otimes^{\mathbb{L}}_A k \cong \Big(k \otimes^{\mathbb{L}}_A k\Big)^{\otimes 2}.$$

The construction (105) can be rewritten as

$$Bar(A) \;\cong\; k \overset{\mathbb{L}}{\underset{A}{\otimes}} \int_I A \overset{\mathbb{L}}{\underset{A}{\otimes}} k$$

using the natural $A \otimes A^{op} \cong \int_{S^0} A$-module structure of $\int_I A$. Now the E_1-coalgebra structure of $Bar(A)$ is given by the inclusion of two disjoint open intervals I_1 and I_2 inside I. We denote J_1, J_2, J_3 the three disjoint intervals whose union is the complement $I \setminus (I_1 \cup I_2)$. Unfolding the definition of the map $\delta_{I_1,I_2,I}$ given

by Definition 8.36 and Lemma 8.35, using excision for factorization homology (see [L-HA, F1, GTZ2, AFT]), we find that, $\delta_{I_1, I_2, I}$ is the composition

$$(125) \quad Bar(A) \cong k \overset{\mathbb{L}}{\underset{A}{\otimes}} \int_I A \overset{\mathbb{L}}{\underset{A}{\otimes}} k \longrightarrow k \overset{\mathbb{L}}{\underset{A}{\otimes}} \int_{J_1} A \overset{\mathbb{L}}{\underset{A}{\otimes}} \int_{I_1} A \overset{\mathbb{L}}{\underset{A}{\otimes}} \int_{J_2} A \overset{\mathbb{L}}{\underset{A}{\otimes}} \int_{I_2} A \overset{\mathbb{L}}{\underset{A}{\otimes}} \int_{J_3} A \overset{\mathbb{L}}{\underset{A}{\otimes}} k$$

$$\overset{id\otimes_A^{\mathbb{L}} \epsilon \otimes_A^{\mathbb{L}} id \otimes_A^{\mathbb{L}} \epsilon \otimes_A^{\mathbb{L}} id \otimes_A^{\mathbb{L}} \epsilon \otimes_A^{\mathbb{L}} id}{\longrightarrow} k \overset{\mathbb{L}}{\underset{A}{\otimes}} \int_{I_1} A \overset{\mathbb{L}}{\underset{A}{\otimes}} k \overset{\mathbb{L}}{\underset{A}{\otimes}} \int_{I_2} A \overset{\mathbb{L}}{\underset{A}{\otimes}} k \cong Bar(A) \otimes Bar(A).$$

Hence, the underlying coproducts of the E_1-coalgebra structure on $Bar(A)$ realize the map (124). Thus, they induce the E_1-coalgebra structure of $Bar^{std}(A)$ under the equivalence given by Lemma 8.13.

We are left to prove Claim (3). By Proposition 8.14 and Lemma 8.17, we know that the iterated bar functor $Bar^{(n)}$ from Theorem 8.37 coincides in $E_\infty\text{-Alg}^{aug}$ with the the one obtained in § 8.1.

We need to compare the E_n-coalgebra structures. By Lemma 6.16, we are left to compare the structure maps $\delta_{c_1,\ldots,c_r,\mathbb{R}^n} : \bigotimes_{i=1}^r Bar^{(n)}(A)(c_i) \to Bar^{(n)}(A)(\mathbb{R}^n)$ with the maps (99) giving rise to the structure in Theorem 8.9.

Further, from equivalence (122) above, Proposition 8.8 and its proof we obtain a commutative diagram of equivalences

$$(126) \qquad p_*(\widehat{\mathcal{A}_U}) = Bar^{(n)}(A)(U) \overset{\cong}{\longrightarrow} CH_{\widehat{U}}(A) \overset{\mathbb{L}}{\underset{A}{\otimes}} k \cong CH_{\widehat{U}}(A, k)$$

$$\cong \Big\uparrow \qquad\qquad\qquad\qquad \Big\uparrow \cong$$

$$\int_U A \underset{\int_{U\setminus\{*_U\}} A}{\overset{\mathbb{L}}{\otimes}} k \overset{\cong}{\longrightarrow} CH_U(A) \underset{CH_{U\setminus\{*_U\}}(A)}{\overset{\mathbb{L}}{\otimes}} k$$

for every convex open set (in particular cube) $U \subset \mathbb{R}^n$. The lower arrow of the diagram is the tensor product of the equivalences between factorization and Hochschild homology given by Theorem 3.13.

We wish to analyze the structure maps $\delta_{U_1,\ldots,U_r,V}$ (where all the sets U_i's, V are convex) under this equivalence. For any $i = 1\ldots r$, from the above diagram (126) and the definition of the map (117), we get the commutative diagrams

$$(127) \qquad\qquad p_*(\widehat{\mathcal{A}_V}) \overset{\tilde{\gamma}_{U_i}^V}{\longrightarrow} p_*(\widehat{\mathcal{A}_{U_i}})$$

$$\cong \Big\downarrow \qquad\qquad\qquad \Big\downarrow \cong$$

$$CH_{\widehat{V}}(A) \overset{\mathbb{L}}{\underset{A}{\otimes}} k \overset{(\iota_{U_i}^V)_* \otimes id}{\longrightarrow} CH_{\widehat{U_i}}(A) \overset{\mathbb{L}}{\underset{A}{\otimes}} k$$

where $\iota_{U_i}^V : \widehat{V} \to \widehat{U_i}$ is the map (113) which collapses the complement of U_i in $\widehat{V}$ to a point.

Recall that $\delta_{U_1,\ldots,U_r,V}$ is the tensor product $\bigotimes \tilde{\gamma}_{U_i}^V$ (Definition 8.36), tensoring the commutative diagrams (127) applied to cubes $U_1,\ldots,U_r$ inside $V = \mathbb{R}^n$, we get

the commutative diagram

$$
\begin{array}{ccc}
Bar^{(n)}(A)(\mathbb{R}^n) = p_*(\widehat{\mathcal{A}_{\mathbb{R}^n}}) & \xrightarrow{\delta_{U_1,\ldots,U_r,\mathbb{R}^n}} & \bigotimes_{i=1}^{r} p_*(\widehat{\mathcal{A}_{U_i}}) = \bigotimes_{i=1}^{r} Bar^{(n)}(A)(U_i) \\
\cong \downarrow & & \downarrow \cong \\
CH_{S^n}(A) \overset{\mathbb{L}}{\underset{A}{\otimes}} k & \xrightarrow{pinch_*^{S^n,r}(U_1,\ldots,U_r)} & \bigotimes_{i=1}^{r} \left(CH_{\widehat{U_i}}(A) \overset{\mathbb{L}}{\underset{A}{\otimes}} k \right) \cong \left(CH_{S^n}(A,k) \right)^{\otimes r}
\end{array}
$$

where the lower map is the pinching map (99) applied to the cubes $U_1, \ldots, U_r$. Together with Lemma 6.16, this proves that the E_n-coalgebra structure given by Theorem 8.9 is the same as the one from Theorem 8.37. □

Remark 8.41 (E_n**-analogues of (homotopy) bialgebras**). The category of (differential graded) bialgebras is the same as the category coAlg(Alg) of (differential graded) coalgebra objects in the category of (differential graded) algebras.

In particular, the $(\infty, 1)$-category E_1-coAlg$(E_1$-Alg$)$ is equivalent to the $(\infty, 1)$-category of (differential graded) bialgebras in k-Mod$_\infty$.

We thus think of E_p-coAlg$(E_q$-Alg$)$ as analogues of bialgebras with some commutativity and cocommutativity conditions lying in between (dg-)bialgebras and (dg-)commutative and cocommutative bialgebras.

Note that in characteristic zero, by choice of a formality isomorphism $\mathbb{P}_d \cong \mathbb{E}_d$, a model for the ∞-category E_d-coAlg$(E_1$-Alg$)$ is given by the ∞-category of (homotopy) d-bialgebras considered by Tamarkin [Ta2].

Also, Proposition 8.37 implies that the Bar construction of an E_2-algebra is naturally a (homotopy) bialgebra. It would be interesting to relate this result with a "somehow dual" result of Kadeishvili [Ka] stating that the cobar construction of a (dg-)bialgebra has an natural structure of homotopy Gerstenhaber algebra structure, hence of E_2-algebras in characteristic zero.

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