CUBICAL RIGIDIFICATION, THE COBAR CONSTRUCTION, AND THE BASED LOOP SPACE

MANUEL RIVERA AND MAHMOUD ZEINALIAN

Abstract. We define a functor $\mathfrak{C}_{\square_c}$ from simplicial sets to categories enriched over cubical sets with connections which, after triangulating mapping spaces, induces Lurie's rigidification functor $\mathfrak{C}$ which associates a simplicial category to any simplicial set. We prove that taking normalized chains on the mapping spaces of $\mathfrak{C}_{\square_c}$ yields a functor Λ from simplicial sets to differential graded categories which is left adjoint to the differential graded nerve functor. Furthermore, we show that for any pointed connected topological space (X, b), the differential graded associative algebra $\Lambda(\mathrm{Sing}(X, b))(b, b)$ models the based loop space of X at b, where $\mathrm{Sing}(X, b)$ is the 0-reduced singular simplicial set of X. We show $\Lambda(\mathrm{Sing}(X, b))(b, b)$ is isomorphic to the cobar construction of the differential graded coassociative algebra of normalized chains in X, extending Adams' original result to possible non-simply connected spaces.

1. Introduction

In order to compare two different models for ∞-categories, Lurie constructs in [Lur09] a *rigidification* functor $\mathfrak{C} : Set_\Delta \to Cat_\Delta$, where Set_Δ denotes the category of simplicial sets and Cat_Δ the category of simplicial categories (categories enriched over simplicial sets). For a standard n-simplex Δ^n the simplicial category $\mathfrak{C}(\Delta^n)$ has the set $[n] = \{0, 1, ..., n\}$ as objects and for any $i, j \in [n]$ with $i \leq j$ the mapping space $\mathfrak{C}(\Delta^n)(i, j)$ is isomorphic to the simplicial cube $(\Delta^1)^{\times j-i-1}$. In particular $\mathfrak{C}(\Delta^n)(0, n) \cong (\Delta^1)^{\times n-1}$ and we think of this simplicial $(n-1)$-cube as parametrizing a family of paths in Δ^n from 0 to n. Adams described in [Ada52] an algebraic construction, called the cobar construction, which when applied to a suitable differential graded coassociative coalgebra model of a simply connected space X produces a differential graded associative algebra (dga) model for the based loop space of X. Adams' construction is based on certain geometric maps $\theta_n : I^{n-1} \to P_{0,n}|\Delta^n|$, where $P_{0,n}|\Delta^n|$ is the space of paths in the topological n-simplex $|\Delta^n|$ from vertex 0 to vertex n, satisfying a compatibility equation that relates the cubical boundary to the simplicial face maps and the Alexander-Whitney coproduct. The definition of $\mathfrak{C}(\Delta^n)(0, n)$ resembles the construction of Adams' maps θ_n.

In this article we describe explicitly the relationship between Lurie's functor $\mathfrak{C}$ and Adams' cobar construction. To achieve this, we factor the functor $\mathfrak{C}$ through a functor $\mathfrak{C}_{\square_c}$ from simplicial sets to categories enriched over cubical sets with connections. We show that the functor Λ from simplicial sets to dg-categories defined by applying the normalized chains functor to the mapping spaces of $\mathfrak{C}_{\square_c}$ is the left adjoint of the differential graded nerve functor described in [Lur11]. Furthermore, when Λ is applied to $\mathrm{Sing}(X, b)$, the 0-reduced simplicial set of a pointed connected topological space (X, b) consisting of singular simplices on X with vertices at b, we obtain a dga weakly equivalent to the dga of singular chains on $\Omega_b X$, the Moore based loop space of X at b. Moreover, we prove $\Lambda(\mathrm{Sing}(X,b))(b,b)$ is isomorphic to the cobar construction of the differential graded coassociative coalgebra of normalized chains on $\mathrm{Sing}(X,b)$. These statements do not assume X is simply connected and therefore extend Adams' original result.

The functor $\mathfrak{C}$ is first defined on standard simplices and then extended to general simplicial sets as a left Kan extension. For any simplicial set S, Dugger and Spivak computed in [DS11] the mapping spaces $\mathfrak{C}(S)(x,y)$ in terms of necklaces. A necklace is a simplicial set of the form $T = \Delta^{n_1} \vee ... \vee \Delta^{n_k}$; a necklace in S from x to y is a map of simplicial sets $f : T \to S$, where T is a necklace, and f sends the first vertex of T to x and the last vertex of T to y. For any necklace T one may associate functorially a simplicial cube $C(T)$ and one of the main results in [DS11] is that $\mathfrak{C}(S)(x,y)$ is isomorphic to the colimit of the simplicial sets $C(T)$ over necklaces T in S from x to y. It is tempting to replace the simplicial cubes $C(T)$ with standard cubical sets of the same dimension to obtain a cubical version of $\mathfrak{C}$. However, there are certain maps between necklaces that do not correspond to cubical maps. For example the codegeneracy map $s^1 : \Delta^3 \to \Delta^2$ which collapses the edge [12] in Δ^3 yields a map between simplicial cubes $C(s^1) : C(\Delta^3) \to C(\Delta^2)$ which does not correspond to a codegeneracy map between standard cubical sets. Nonetheless, $C(s^1)$ corresponds to a *co-connection* morphism, whose definition is recalled in Section 2. Cubical sets with connections were introduced in [BH81] and can be thought of as cubical sets with extra degeneracies. In Section 3 we describe explicitly the morphisms in the category of necklaces and then in Section 4 we explain how cubical sets with connections arise naturally from necklaces. We use the results in Sections 3 and 4 and the description of $\mathfrak{C}(S)(x,y)$ in terms of necklaces to define $\mathfrak{C}_{\square_c}$ in Section 5. In Section 6 we explain how $\mathfrak{C}_{\square_c}$ gives rise to the functor Λ which is the left adjoint of the dg nerve functor. Finally, in Section 7 we describe how Λ produces a dga model for the based loop space of a connected topological space and how it relates to the cobar construction.

We have a Quillen equivalence between model categories given by $\mathfrak{C} : Set_\Delta \to Cat_\Delta$ and the homotopy coherent nerve $N_\Delta : Cat_\Delta \to Set_\Delta$, where Set_Δ is endowed with the Joyal model structure and Cat_Δ with the Bergner model structure. These are model structures such that the fibrant objects are models for ∞-categories. It follows from [Cis06] and [Mal09] that the category of cubical sets with connections is the category of presheaves over a test category and therefore it admits model category structure making it a model for homotopy types. This implies that there is a model category structure on categories enriched over cubical sets with connections making it a model for ∞-categories. In order to show $\mathfrak{C}_{\square_c}$ is part of a Quillen equivalence, a more concrete study of the model structure given by the main result of [Cis06] on cubical sets with connection is needed. Namely, we must show that the triangulation functor from cubical sets with connections (with the model structure obtained from [Cis06]) to simplicial sets (with Quillen model structure) together with the cubical singular functor with connections going in the other direction give a Quillen equivalence. This should follow from arguments similar to those in [Jar06] and we will take this up in subsequent work.

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2. Preliminaries

Denote by Set the category of sets. For any small category $\mathcal{C}$ denote by $Set_{\mathcal{C}}$ the category of presheaves on $\mathcal{C}$ with values in Set, so the objects of $Set_{\mathcal{C}}$ are functors $\mathcal{C}^{op} \to Set$ and morphisms are natural transformations between them. For example, if Δ is the category of non-empty finite ordinals with order preserving maps then Set_Δ is the category of *simplicial sets*. We denote by Δ^n the *standard n-simplex*, so Δ^n is obtained by applying the Yoneda emedding to $[n]$, namely $\Delta^n : [m] \mapsto \mathrm{Hom}_\Delta([m],[n])$. Recall that morphisms in the category Δ are generated by functions of two types: cofaces $d_i : [n] \to [n+1]$, $0 \leq i \leq n+1$, and codegeneracies $s_j : [n] \to [n-1]$, $0 \leq j \leq n-1$. The Yoneda embedding yields simplicial set morphisms between standard simplices $Y(d_i) : \Delta^n \to \Delta^{n+1}$ and $Y(s_j) : \Delta^n \to \Delta^{n-1}$ which we call *coface* and *codegeneracy (simplicial) morphisms*. We say a simplicial set S is 0-reduced if the set S_0 is a singleton.

For any positive integer n, let $\mathbf{1}^n$ be the n-fold cartesian product of copies of the category $\mathbf{1} = \{0,1\}$ which has two objects and one non-identity morphism. Denote by $\mathbf{1}^0$ the category with one object and one morphism. We will consider presheaves over the category $\square_c$ which is defined as follows. The objects of $\square_c$ are the categories $\mathbf{1}^n$ for $n = 0,1,2,\ldots$. The morphisms in $\square_c$ are generated by functors of the following three kinds:
cubical co-face functors $\delta^\epsilon_{j,n} : \mathbf{1}^n \to \mathbf{1}^{n+1}$, where $j = 0,1,\ldots,n+1$, and $\epsilon \in \{0,1\}$, defined by

$$\delta^\epsilon_{j,n}(s_1,\ldots,s_n) = (s_1,\ldots,s_{j-1},\epsilon,s_j,\ldots,s_n),$$

cubical co-degeneracy functors $\varepsilon_{j,n} : \mathbf{1}^n \to \mathbf{1}^{n-1}$, where $j = 1,\ldots,n$, defined by

$$\varepsilon_{j,n}(s_1,\ldots,s_n) = (s_1,\ldots,s_{j-1},s_{j+1},\ldots,s_n), \text{ and}$$

cubical co-connection functors $\gamma_{j,n} : \mathbf{1}^n \to \mathbf{1}^{n-1}$, where $j = 1,\ldots,n-1$, $n \geq 2$, defined by

$$\gamma_{j,n}(s_1,\ldots,s_n) = (s_1,\ldots,s_{i-1},\max(s_i,s_{i+1}),s_{i+2},\ldots,s_n).$$

Objects in the category $Set_{\square_c}$ are called *cubical sets with connections* and were introduced by Brown and Higgins in [BH81]. For any cubical set with connections K we have a collection of sets $\{K_n := K(\mathbf{1}^n)\}_{n\in\mathbb{Z}_{\geq 0}}$ together with *cubical face maps* $\partial^\epsilon_{j,n} := K(\delta^\epsilon_{j,n}) : K_{n+1} \to K_n$, *cubical degeneracy maps* $E_{j,n} := K(\varepsilon_{j,n}) : K_{n-1} \to K_n$, and *connections* $\Gamma_{j,n} := K(\gamma_{j,n}) : K_{n-1} \to K_n$. For simplicity we often drop the second index in this notation and for example write ∂_j instead of $\partial_{j,n}$. Elements of K_n are called n-cells. The structure maps satisfy certain compatibilities described in [BH81]. The *standard n-cube with connections* $\square^n_c$ is the presheaf on $\square_c$ represented by $\mathbf{1}^n$, namely $\mathrm{Hom}_{\square_c}(\_\,,\mathbf{1}^n) : \square_c^{op} \to Set$.

For a fixed commutative unital ring k denote by Ch_k the category of positively graded chain complexes over k. The tensor product over k defines on Ch_k a symmetric monoidal structure. We have normalized Moore chains functors $Q_\Delta : Set_\Delta \to$

Ch_k and $Q_{\square_c} : Set_{\square_c} \to Ch_k$. The definition of Q_Δ is standard; we recall the definition of $Q_{\square_c}$. First let C_*K be the chain complex such that C_nK is the free k-module generated by elements of K_n with differential $\partial : K_n \to K_{n-1}$ defined on $\sigma \in K_n$ by $\partial(\sigma) := \sum_{j+1}^{n}(-1)^j(\partial^1_{j,n-1}(\sigma) - \partial^0_{j,n-1}(\sigma))$. Let D_nK be the submodule of C_nK which is generated by those cells in K_n which are the image of a degeneracy or of a connection map $K_{n-1} \to K_n$. The graded module D_*K forms a subcomplex of C_*K. Define $Q_{\square_c}(K)$ to be the quotient chain complex C_*K/D_*K.

The category Set_Δ has a symmetric monoidal structure given by the cartesian product of simplicial sets. We will use the following (non-symmetric) monoidal structure on $Set_{\square_c}$: for cubical sets with connections K and K' define

$$K \otimes K' := \underset{\sigma:\square^n_c \to K, \tau:\square^n_c \to K'}{\operatorname{colim}} \square_c^{n+m}.$$

Using the above monoidal structures we may define Cat_Δ the category of small categories enriched over simplicial sets; these are called *simplicial categories*. Similarly denote by $Cat_{\square_c}$ the category of small categories enriched over cubical sets with connections; these are called *cubical categories with connections*. We will also consider the category $dgCat_k$ of small categories enriched over chain complexes over k; these are called *dg categories*.

3. The category of necklaces

A *necklace* T is a simplicial set of the form $T = \Delta^{n_1} \vee ... \vee \Delta^{n_k}$ where $n_i \geq 0$. A morphism $f : T \to T'$ of necklaces is a map of simplicial sets which preserves the first and last vertices. We say a necklace $\Delta^{n_1} \vee ... \vee \Delta^{n_k}$ is of *preferred form* if $k = 0$ or each $n_i \geq 1$. Let $T = \Delta^{n_1} \vee ... \vee \Delta^{n_k}$ be a necklace in preferred form. Each Δ^{n_i} is called a *bead* of T. Denote by b_T the number of beads in T. A *joint* of T is either an initial or a final vertex in some bead. Given a necklace T write V_T and J_T for the sets of vertices and joints of T, respectively. For any two vertices $a, b \in V_T$ we write $V_T(a, b)$ and $J_T(a, b)$ for the set of vertices and joints between a and b inclusive. Note that there is a unique subnecklace $T(a, b) \subseteq T$ with joints $J_T(a, b)$ and vertices $V_T(a, b)$. Denote by Nec the category whose objects are necklaces in preferred form and morphisms are morphisms of necklaces. Note that Nec is a full subcategory of $Set^{*,*}_\Delta = \partial\Delta^1 \downarrow Set_\Delta$.

Proposition 3.1. *Any non-identity morphism in Nec is a composition of morphisms of the following type*

(i) $f : T \to T'$ is an injective morphism of necklaces and $|V_{T'} - J_{T'}| - |V_T - J_T| = 1$

(ii) $f : \Delta^{n_1} \vee ... \vee \Delta^{n_k} \to \Delta^{m_1} \vee ... \vee \Delta^{m_k}$ is a morphism of necklaces of the form $f = f_1 \vee ... \vee f_k$ such that for exactly one p, $f_p : \Delta^{n_p} \to \Delta^{m_p}$ is a codegeneracy morphism (so $m_p = n_p - 1$) and for all $i \neq p$, $f_i : \Delta^{n_i} \to \Delta^{m_i}$ is the identity map of standard simplices (so $n_i = m_i$ for $i \neq p$)

(iii) $f : \Delta^{n_1} \vee ... \vee \Delta^{n_{p-1}} \vee \Delta^1 \vee \Delta^{n_{p+1}} \vee ... \vee \Delta^{n_k} \to \Delta^{n_1} \vee ... \vee \Delta^{n_{p-1}} \vee \Delta^{n_{p+1}} \vee ... \vee \Delta^{n_k}$ is a morphism of necklaces such that f collapses the p-th bead Δ^1 in the domain to the last vertex of the $(p-1)$-th bead in the target and the restriction of f to all the other beads is injective.

Proof. We prove that any non-identity morphism of necklaces $f : T \to T'$ is a composition of morphisms of type (i), (ii), and (iii) by induction on b_T, the number of beads of T. If $b_T = 1$, then we must have $b_{T'} = 1$ as well, so f is a morphism of simplicial sets between standard simplices which preserves first and last vertices. It follows that f is a composition of (simplicial) coface and codegeneracy morphisms. Cofaces and codegeneracies between standard simplices are morphisms of necklaces of type (i) and of type (ii) or (iii), respectively. Assume we have shown the proposition for $b_T \leq k$ and suppose $b_T = k + 1$. Let $V_T = \{x_0, ..., x_p\}$ be the vertices of T and $x_i \preceq x_{i+1}$. Let x_{j_0} be the last vertex of the first bead of T, so $T = T(x_0, x_{j_0}) \vee T(x_{j_0}, x_p)$ where $T(x_0, x_{j_0})$ has one bead and $T(x_{j_0}, x_p)$ has k beads. Let $T_f = T'(f(x_0), f(x_{j_0})) \vee T'(f(x_{j_0}), f(x_p))$. We have an injective morphism of necklaces $t : T_f \to T'$ (notice that it is possible for $T_f \neq T'$ since $f(x_{j_0})$ might not be a joint of T'). It follows that $f = t \circ (g \vee h)$ where $g : T(x_0, x_{j_0}) \to T'(f(x_0), f(x_{j_0}))$ and $h : T(x_{j_0}, x_p) \to T'(f(x_{j_0}), f(x_p))$ are the morphisms of necklaces induced by restricting f to $T(x_0, x_{j_0})$ and $T(x_{j_0}, x_p)$ respectively. By the induction hypothesis each of g and h is a composition of morphisms of type (i), (ii), and (iii) and this implies that $g \vee h$ is a composition of such morphisms as well. In fact, we have

$$g \vee h = (id_{T'(f(x_0), f(x_{j_0}))} \vee h) \circ (g \vee id_{T(x_{j_0}, x_p)})$$

and, clearly, the wedge of an identity morphism and a morphism which is a composition of morphisms of type (i), (ii), and (iii) is again a morphism of such form.

To conclude the proof we show that $t : T_f \to T'$ is of the desired form. More generally, let us prove that any non-identity injective morphism of necklaces $t : R \to R'$ is a composition of morphisms of type (i) by induction on the integer $l(R, R') := |V_{R'} - J_{R'}| - |V_R - J_R|$. If $l(R, R') = 1$ then t is of type (i). Assume we have shown the claim for $l(R, R') = k$. Suppose $t : R \to R'$ is injective and $l(R, R') = k + 1$, then we have two cases: either (a) $J_{R'} = t(J_R)$ or (b) $J_{R'} \subset t(J_R)$. In case (a), it follows that both R and R' have the same number of beads, thus $t = i \circ j$ for inclusions of necklaces $j : R \to S$, $i : S \to R'$ where S is the subnecklace of R' spanned by $t(V_R) \cup \{v\}$ and v is the smallest element of $V_{R'} - t(V_R)$. Then j is of type (i) and i is a composition of morphisms of type (i) by the induction hypothesis. For case (b), let $t(J_R) - J_{R'} = \{t(x_{i_1}), ..., t(x_{i_n})\}$ and consider the unique subnecklace S of R' defined by $V_S = t(V_R)$ and $J_S = t(J_R) - \{t(x_{i_1})\}$. Then we have $t = i \circ j$ for inclusions of necklaces $j : R \to S$, $i : S \to R'$ with j of type (i) and i a composition of type (i) morphisms by the induction hypothesis. □

Remark 3.2. Let us consider type (i) morphisms of the form $f : T \to \Delta^p$ for some integer $p \geq 1$. If $b_T = 1$ then we have an injective map of simplicial sets $f : \Delta^{p-1} \to \Delta^p$ which sends the first (resp. last) vertex of Δ^{p-1} to the first (resp. last) vertex of Δ^p. The morphism f determines a $(p-1)$-simplex of the simplicial set Δ^p, i.e. an element of $(\Delta^p)_{p-1}$. There are $p+1$ non-degenerate elements in $(\Delta^p)_{p-1}$, however only $p-1$ of these can correspond to f based on the constraint that f must preserve first and last vertices, namely, all the faces of the unique non-degenerate element in $(\Delta^p)_p$ except the first and last. If $b_T > 1$ then there is a joint $v \in J_T$ such that $f(v) \notin J_{T'}$. Moreover, since f is injective and $|V_{T'} - J_{T'}| - |V_T - J_T| = 1$, we have $f(J_T - \{v\}) = J'_T$ and $f(V_T) = V_{T'}$. It follows that $b_T = 2$ and the image of f is a subnecklace $T'_1 \vee T'_2$ of Δ^p starting and ending with the first and last vertices of Δ^p, respectively, and containing all the vertices of Δ^p. Hence, we have $T'_1 \vee T'_2 = \Delta^{p-i} \vee \Delta^i$ for some $0 < i < p$ and each of these subnecklaces of Δ^p

corresponds to a unique term in the formula for the Alexander-Whitney diagonal $Q_\Delta(\Delta^p) \to Q_\Delta(\Delta^p) \otimes Q_\Delta(\Delta^p)$ applied to the generator represented by the unique non-degenerate p-simplex in $(\Delta^p)_p$.

4. The functor $C_{\square_c} : Nec \to Set_{\square_c}$

The goal of this section is to define a functor $C_{\square_c} : Nec \to Set_{\square_c}$. We start by defining a functor $P : Nec \to Cat$ where Cat is the category of small categories. Given a necklace T and two vertices $a, b \in V_T$ we may define a small category $P_T(a,b)$ whose objects are subsets $X \subseteq V_T(a,b)$ such that $J_T(a,b) \subseteq X$ and morphisms are inclusions of sets. For any necklace $T \in Nec$ let $P(T) = P_T(\alpha, \omega)$ where $\alpha, \omega \in V_T$ are the first and last vertices of T. Let $f : T \to T'$ be a morphism in Nec, so f is a map of simplicial sets such that $f(\alpha) = \alpha'$ and $f(\omega) = \omega'$ where $\alpha, \omega \in V_T$ and $\alpha', \omega' \in V_{T'}$ are the first and last vertices of T and T', respectively. Notice that we have an inclusion $J_{T'} \subseteq f(J_T)$. Thus f induces a functor $P_f : P_T(\alpha,\omega) \to P_{T'}(\alpha',\omega')$ defined on objects by $P_f(X) = f(X)$ and on morphisms by the induced inclusion of sets. This yields a functor $P : Nec \to Cat$. We might think of the objects of $P(T)$ as strings of 0's and 1's as discussed below. This interpretation will yield a functor $P_{\mathbf{1}}$ which is naturally isomorphic to P. We define a total order on the vertices of a necklace by setting $a \preceq b$ if there is a directed path from a to b.

Proposition 4.1. *For any necklace T and any $a, b \in V_T$ such that $a \preceq b$, there is an isomorphism of categories $\phi_T : P_T(a,b) \cong \mathbf{1}^N$ where $N = |V_T(a,b) - J_T(a,b)|$.*

Proof. Let $V_T(a,b) - J_T(a,b) = \{y_1, ..., y_N\}$ and $y_i \preceq y_{i+1}$ for $i = 1, ..., N-1$. Given any object X of $P_T(a,b)$ (so $J_T(a,b) \subseteq X \subseteq V_T(a,b)$) we define $\phi_T(X) := (\phi_T^1(X), ..., \phi_T^N(X))$ to be the object in the category $\mathbf{1}^N$ where, for $1 \leq i \leq N$, we have $\phi_T^i(X) = 1$ if $y_i \in X$ and $\phi_T^i(X) = 0$ if $y_i \notin X$. Given a morphism $f : X \to Y$ in $P_T(a,b)$ (so f is an inclusion of sets) we have an induced morphism $\phi_T(f) : \phi_T(X) \to \phi_T(Y)$ defined by $\phi_T(f) := (\phi_T^1(f), ..., \phi_T^N(f))$ where, for $1 \leq i \leq N$, $\phi_T^i(f) : \phi_T^i(X) \to \phi_T^i(Y)$ is the unique non-identity morphism in $\mathbf{1}$ if $\phi_T^i(X) = 0$ and $\phi_T^i(Y) = 1$, and $\phi_T^i(f)$ is an identity morphism otherwise. It is clear that the functor $\phi_T : P_T(a,b) \to \mathbf{1}^N$ is an isomorphism of categories. $\square$

Consider the functor $P_{\mathbf{1}} : Nec \to Cat$ defined on objects by $P_{\mathbf{1}}(T) = \mathbf{1}^{|V_T - J_T|}$ and on morphisms $f : T \to T'$ by $P_{\mathbf{1}}(f) = \phi_{T'} \circ P(f) \circ \phi_T^{-1} : \mathbf{1}^{|V_T - J_T|} \to \mathbf{1}^{|V_{T'} - J_{T'}|}$. The above proposition implies that $P_{\mathbf{1}}$ is naturally isomorphic to P. In the following proposition we describe explicitly the functor $P_{\mathbf{1}}(f)$ for morphisms $f : T \to T'$ of type (i), (ii), and (iii).

Proposition 4.2. *Let $f : T \to T'$ be a morphism in Nec and let $N = |V_T - J_T|$.*

(1) *If f is of type (i) then $P_{\mathbf{1}}(f) : \mathbf{1}^N \to \mathbf{1}^{N+1}$ is a cubical co-face functor.*
(2) *If f is of type (ii) then $P_{\mathbf{1}}(f) : \mathbf{1}^N \to \mathbf{1}^{N-1}$ is either a cubical co-connection functor or a cubical co-degeneracy functor.*
(3) *If f is of type (iii) then $P_{\mathbf{1}}(f) : \mathbf{1}^N \to \mathbf{1}^N$ is the identity functor.*

Proof. For any morphism of necklaces $f : T \to T'$ we have $J_{T'} \subseteq f(J_T)$. For $f : T \to T'$ of type (i) we prove below that if $J_{T'} \subset f(J_T)$ then $P_{\mathbf{1}}(T)(f)$ is a cubical co-face functor $\delta^1_{j,N}$ and if $J_{T'} = f(J_T)$ then $P_{\mathbf{1}}(T)(f)$ is a cubical co-face functor $\delta^0_{j,N}$. A morphism $f : T \to T'$ of type (ii) collapses two vertices v and w of T into a vertex v' of T' and is injective on $V_T - \{v, w\}$. We prove below that if $v' \notin J_{T'}$ then

$P_{\mathbf{1}}(T)(f)$ is a cubical co-connection functor $\gamma_{j,N}$ and if $v' \in J_{T'}$ then $P_{\mathbf{1}}(T)(f)$ is a cubical co-degeneracy functor $\varepsilon_{j,N}$. The proof for the third part of the proposition will be straightforward.

(1) Let $f : T \to T'$ be of type (i) and write $\{y'_1, ..., y'_{N+1}\} = V_{T'} - J_{T'}$ where $y'_i \preceq y'_{i+1}$. We have $J_{T'} \subseteq f(J_T)$ since f is a morphism of necklaces. If $J_{T'} \subset f(J_T)$ then there is $v \in J_T$ such that $f(v) = y'_j \in V_{T'} - J_{T'}$ for some $j \in \{1, ..., N+1\}$ and $f(J_T - \{v\}) \subseteq J'_T$. Then for any object X in $P(T)$, $v \in J_T \subseteq X$ so $y_j = f(v) \in f(X)$. Using the fact that f is injective and identifying objects X in $P(T)$ with sequences of 0's and 1's via the isomorphism $\phi_T : P(T) \cong \mathbf{1}^N$ we see that $P_{\mathbf{1}}(f) : \mathbf{1}^N \to \mathbf{1}^{N+1}$ is given on objects by

$$P_{\mathbf{1}}(f)(s_1, ..., s_N) = (s_1, ..., s_{j-1}, 1, s_j, ..., s_N)$$

and on morphisms $\lambda = (\lambda_1, ..., \lambda_N) : (s_1, ..., s_N) \to (s'_1, ..., s'_N)$ by

$$P_{\mathbf{1}}(f)(\lambda) = (\lambda_1, ..., \lambda_{j-1}, id_1, \lambda_j, ..., \lambda_N).$$

Thus $P_{\mathbf{1}}(f)$ is the cubical co-face functor $\delta^1_{j,N}$.

If $J_{T'} = f(J_T)$ then there exists exactly one $j \in \{1, ..., N+1\}$ such that $f^{-1}(y'_j) = \emptyset$. Then for any object X in $P(T)$, y'_j will never be an element of $f(X)$. Using the fact that f is injective and identifying objects X in $P(T)$ with sequences of 0's and 1's via the isomorphism $\phi_T : P(T) \cong \mathbf{1}^N$ we see that $P_{\mathbf{1}}(f) : \mathbf{1}^N \to \mathbf{1}^{N+1}$ is given on objects by

$$P_{\mathbf{1}}(f)(s_1, ..., s_N) = (s_1, ..., s_{j-1}, 0, s_j, ..., s_N)$$

and on morphisms $\lambda = (\lambda_1, ..., \lambda_N) : (s_1, ..., s_N) \to (s'_1, ..., s'_N)$ by

$$P_{\mathbf{1}}(f)(\lambda) = (\lambda_1, ..., \lambda_{j-1}, id_0, \lambda_j, ..., \lambda_N).$$

It follows that $P_{\mathbf{1}}(f)$ is the cubical co-face functor $\delta^0_{j,N}$.

(2) Let $f : T \to T'$ be of type (ii) and write $\{y_1, ..., y_N\} = V_T - J_T$ where $y_i \preceq y_{i+1}$ and $\{y'_1, ..., y'_{N-1}\} = V_{T'} - J_{T'}$ where $y'_i \preceq y'_{i+1}$. There exists $v' \in V_{T'}$ such that $f^{-1}(v') = \{v, w\}$ for some $v, w \in V_T$ and $|f^{-1}(x')| = 1$ for all $x' \in V_{T'} - \{v'\}$. Note that v and w are consecutive vertices in the p-th bead of T. We have two cases: either $v' \in V_{T'} - J_{T'}$ or $v' \in J_{T'}$.

If $v' \in V_{T'} - J_{T'}$, then $v, w \in V_T - J_T$ so we may write $v = y_j$ and $w = y_{j+1}$ for some $j \in \{1, ..., N-1\}$. Hence, for any object X of $P(T)$ we have that if $X \cap \{y_j, y_{j+1}\} \neq \emptyset$ then $v' \in f(X)$ and if $X \cap \{y_j, y_{j+1}\} = \emptyset$ then $v' \notin f(X)$. By identifying objects X in $P(T)$ with sequences of 0's and 1's via the isomorphism $\phi_T : P(T) \cong \mathbf{1}^N$ we see that $P_{\mathbf{1}}(f) : \mathbf{1}^N \to \mathbf{1}^{N-1}$ is given on objects by

$$P_{\mathbf{1}}(f)(s_1, ..., s_N) = (s_1, ...s_{j-1}, max(s_j, s_{j+1}), s_{j+2}, ..., s_N)$$

and on morphisms $\lambda = (\lambda_1, ..., \lambda_N) : (s_1, ..., s_N) \to (s'_1, ..., s'_N)$ by

$$P_{\mathbf{1}}(f)(\lambda) = (\lambda_1, ..., \lambda_{j-1}, \sigma_{j,j+1}, \lambda_{j+2}, ..., \lambda_N),$$

where $\sigma_{j,j+1}$ is the unique morphism $max(s_j, s_{j+1}) \to max(s'_j, s'_{j+1})$ in the category $\mathbf{1}$. It follows that $P_{\mathbf{1}}(f)$ is the cubical co-connection functor $\gamma_{j,N}$.

If $v' \in J_{T'}$, we may assume without loss of generality that $w \in J_T$ and

$v = y_j \in V_T - J_T$ for some $j \in \{1, ..., N\}$. Let X be any object of $P(T)$. Every element of $X - \{y_j\}$ corresponds to a unique element in $f(X)$ via $P(f)$ (since f is of type (ii)) and if $y_j \in X$ then $P(f)$ sends y_j to the joint $v' \in f(X)$. By identifying objects X in $P(T)$ with sequences of 0's and 1's via the isomorphism $\phi : P(T) \cong \mathbf{1}^N$ we see that $P_{\mathbf{1}}(f) : \mathbf{1}^N \to \mathbf{1}^{N-1}$ is given on objects by

$$P_{\mathbf{1}}(f)(s_1, ..., s_N) = (s_1, ..., s_{j-1}, s_{j+1}, ..., s_N)$$

and on morphisms $\lambda = (\lambda_1, ..., \lambda_N) : (s_1, ..., s_N) \to (s'_1, ..., s'_N)$ by

$$P_{\mathbf{1}}(f)(\lambda) = (\lambda_1, ..., \lambda_{i-1}, \lambda_{i+1}, ..., \lambda_N).$$

It follows that $P_{\mathbf{1}}(f)$ is the cubical co-degeneracy functor $\varepsilon_{j,N}$.

(3) If f is of type (iii) then $|V_T| = |V_T| + 1$ and the injectivity of f only fails when it collapses two joints (the endpoints of the p-th bead Δ^1) to a joint in T'. Under the isomorphism $\phi_T : P(T) \cong \mathbf{1}^N$ this collapse does not have any effect since given an object X of $P(T)$ the entries in the string $\phi_T(X)$ of 0's and 1's only indicate which non-joint vertices of T are in X. It follows that $P_{\mathbf{1}}(f) : \mathbf{1}^N \to \mathbf{1}^N$ is the identity functor.

□

Remark 4.3. Consider two morphisms of necklaces $f : U \to T$ and $g : V \to T$. If f and g are both of type (i) and $f \neq g$ then $P_{\mathbf{1}}(f) \neq P_{\mathbf{1}}(g)$. If f and g are of both of type (ii) and $f \neq g$ we may have $P_{\mathbf{1}}(f) = P_{\mathbf{1}}(g)$. For example, let $U = W \vee \Delta^{m+1} \vee \Delta^n \vee W'$, $V = W \vee \Delta^m \vee \Delta^{n+1} \vee W'$, $T = W \vee \Delta^m \vee \Delta^n \vee W'$, for any two necklaces W and W'. Consider the maps $f = id_W \vee s_{m+1} \vee id_{\Delta^n} \vee id_{W'}$ and $g = id_W \vee id_{\Delta^m} \vee s_1 \vee id_{W'}$, where $s_{m+1} : \Delta^{m+1} \to \Delta^m$ and $s_1 : \Delta^{n+1} \to \Delta^n$ are the last and first (simplicial) codegeneracy morphisms respectively. It follows that $P_{\mathbf{1}}(f) = P_{\mathbf{1}}(g)$. The identification of these two morphisms after applying $P_{\mathbf{1}}$ should be compared with the identification in the definition of the Day convolution monoidal structure of the category of cubical sets with connections. Finally, if f and g are of type (iii), then we always have $P_{\mathbf{1}}(f) = P_{\mathbf{1}}(g)$.

Corollary 4.4. *The functor $P_{\mathbf{1}} : Nec \to Cat$ factors as a composition $Nec \to \square_c \hookrightarrow Cat$.*

Proof. For any object T in Nec, $P_{\mathbf{1}}(T) = \mathbf{1}^N$ is an object of $\square_c$ and, by Proposition 4.2, for any morphism f in Nec, $P_{\mathbf{1}}(f)$ is a morphism in $\square_c$. □

Hence, we may consider $P_{\mathbf{1}}$ as a functor from Nec to $\square_c$. Finally, we define a functor from the category of necklaces to the category of cubical sets as follows.

Definition 4.5. Define the functor $C_{\square_c} : Nec \to Set_{\square_c}$ to be the composition of functors $C_{\square_c} := Y \circ P_{\mathbf{1}}$ where $Y : \square_c \to \mathrm{Hom}_{Cat}((\square_c)^{op}, Set) = Set_{\square_c}$ is the Yoneda embedding.

Note that for any T in Nec, $C_{\square_c}(T)$ is the standard cube with connections $\square_c^N$ where $N = |V_T - J_T|$.

Remark 4.6. All non-degenerate cells of $C_{\square_c}(T)$ can be realized as injective maps of necklaces $T' \to T$. More precisely, for every non-degenerate cell $\sigma \in C_{\square_c}(T)_n$ there is a necklace T_σ, with $|V_{T_\sigma} - J_{T_\sigma}| = n$ together with an injective map of necklaces $\iota_\sigma : T_\sigma \to T$ such that the induced map of cubical sets with connections

$$\square_c^n \cong C_{\square_c}(T_\sigma) \xrightarrow{C_{\square_c}(\iota_\sigma)} C_{\square_c}(T)$$

corresponds to the cell σ. Notice T_σ is not unique, since any other T'_σ for which there is a map $T'_\sigma \to T_\sigma$ of type (iii) also works.

5. The functor $\mathfrak{C}_{\square_c} : Set_\Delta \to Cat_{\square_c}$

The goal of this section is to show that the functor $\mathfrak{C} : Set_\Delta \to Cat_\Delta$ defined by Lurie factors naturally through categories enriched over cubical sets with connections via a functor $\mathfrak{C}_{\square_c} : Set_\Delta \to Cat_{\square_c}$. More precisely, we construct functors $\mathfrak{C}_{\square_c} : Set_\Delta \to Cat_{\square_c}$ and $\mathfrak{T} : Cat_{\square_c} \to Cat_\Delta$ such that $\mathfrak{T} \circ \mathfrak{C}_{\square_c}$ is naturally isomorphic to $\mathfrak{C}$.

Definition 5.1. For any simplicial set S we define a category $\mathfrak{C}_{\square_c}(S)$ enriched over cubical sets with connections. Define the objects of $\mathfrak{C}_{\square_c}(S)$ to be the vertices of S, i.e. the elements of S_0. For any $x, y \in S_0$ define

$$\mathfrak{C}_{\square_c}(S)(x,y) := \operatorname*{colim}_{T \to S \in (Nec \downarrow S)_{x,y}} C_{\square_c}(T)$$

where $(Nec \downarrow S)_{x,y}$ is the category whose objects are morphisms $f : T \to S$ for some $T \in Nec$ such that $f(\alpha_T) = x$ and $f(\omega_T) = y$. For any $x, y, z \in S_0$ the composition law

$$\mathfrak{C}_{\square_c}(S)(y,z) \otimes \mathfrak{C}_{\square_c}(S)(x,y) \to \mathfrak{C}_{\square_c}(S)(x,z)$$

is induced as follows. Note that given $T \to S \in (Nec \downarrow S)_{x,y}$ and $U \to S \in (Nec \downarrow S)_{y,z}$, we obtain $T \vee U \to S \in (Nec \downarrow S)_{x,z}$. Then the composition

$$C_{\square_c}(U) \otimes C_{\square_c}(T) \to C_{\square_c}((T \vee U)(\alpha_U, \omega_U)) \otimes C_{\square_c}((T \vee U)(\alpha_T, \omega_T)) \to C_{\square_c}(T \vee U)$$

of morphisms of cubical sets with connections induces the desired composition law after taking colimits. Recall that $(T \vee U)(\alpha_U, \omega_U)$ denotes the unique subnecklace of $T \vee U$ with joints $J_{T\vee U}(\alpha_u, \omega_U)$ and vertices $V_{T \vee U}(\alpha_U, \omega_U)$. It follows from Remark 3.3 that the above composition is well defined, it is, in fact, the map $\square_c^N \otimes \square_c^{N'} \to \square_c^{N+N'}$ of cubical sets with connections. Finally, it is clear that $\mathfrak{C}_{\square_c}(S)$ is functorial in S.

Remark 5.2. The set of n-cells in $\mathfrak{C}_{\square_c}(S)(x,y)$ is

$$\Big(\coprod_{(T \to S) \in (Nec \downarrow S)_{x,y}} C_{\square_c}(T)_n \Big) / \sim$$

where the equivalence relation is generated by $(t : T \to S, \sigma) \sim (t' : T' \to S, \sigma')$ if there is a map of necklaces $f : T \to T'$ such that $t = t' \circ f$ and $C_{\square_c}(f)(\sigma) = \sigma'$. Here $t : T \to S)$ and $t' : T' \to S$ are objects in $(Nec \downarrow S)_{x,y}$, and σ and σ' are n-cells in $C_{\square_c}(T)$ and $C_{\square_c}(T')$, respectively. Any *non-degenerate* n-cell $[t : T \to S, \sigma] \in \mathfrak{C}_{\square_c}(S)(x,y)_n$ may be represented by a pair $(r : R \to S, \sigma_R)$ where R is a necklace with $|V_R - J_R| = n$ such that there are no $(u : U \to S) \in (Nec \downarrow S)_{x,y}$ with $|V_U - J_U| = n - 1$ and $f : R \to U$ satisfying $r = u \circ f$ and $\sigma_R \in C_{\square_c}(R)_n$ is the unique non-degenerate n-cell in $C_{\square_c}(R)$. In fact, one can let $R = T_\sigma$ and $r = t \circ \iota_\sigma$ as in Remark 4.6. These representatives are not unique since we may have another representative $(r' : R' \to S, \sigma_{R'})$ if there is a morphism of necklaces $h : R \to R'$ of type (iii) such that $r' \circ h = r$. We write $[r : R \to S]$ for the equivalence class of the non-degenerate n-cell in $\mathfrak{C}_{\square_c}(S)(x, y)$ represented by $(r : R \to S, \sigma_R)$. Let v be the j-th vertex in $V_R - J_R$. The face map $\partial^1_j : \mathfrak{C}_{\square_c}(S)(x, y)_n \to \mathfrak{C}_{\square_c}(S)(x, y)_{n-1}$ is given by $\partial^1_j[r : R \to S] = [\partial^1_j r : R_v \to S]$ where R_v is the subnecklace of R spanned by vertices $V_R - \{v\}$ and $\partial^1_j r$ is the restriction of r to R_v. The face map $\partial^0_j : \mathfrak{C}_{\square_c}(S)(x, y)_n \to \mathfrak{C}_{\square_c}(S)(x, y)_{n-1}$ is given by $\partial^0_j[r : R \to S] = [\partial^0_j r : R(\alpha_R, v) \vee R(v, \omega_R) \to S]$ where $\partial^0_j r$ is the restriction of r to $R(\alpha_R, v) \vee R(v, \omega_R)$. Of course $[\partial^1_j r : R_v \to S]$ and $[\partial^0_j r : R(\alpha_R, v) \vee R(v, \omega_R) \to S]$ may be degenerate cells in $\mathfrak{C}_{\square_c}(S)(x, y)_{n-1}$ even if $[r : R \to S]$ is non-degenerate.

Let us recall Lurie's construction of $\mathfrak{C} : Set_\Delta \to Cat_\Delta$. Given integers $0 \leq i < j$ we denote by $P_{i,j}$ the category whose objects are subsets of the set $\{i, i+1, ..., j\}$ containing both i and j and morphisms are inclusions of sets. We have an isomorphism of categories $P_{i,j} \cong \mathbf{1}^{j-i-1}$. For each integer $n \geq 0$ define a simplicial category $\mathfrak{C}(\Delta^n)$ whose objects are the elements of the set $\{0, ..., n\}$ and for any two objects i and j such that $i \leq j$, $\mathfrak{C}(\Delta^n)(i, j)$ is the simplicial set $N(P_{i,j})$, where $N : Cat \to Set_\Delta$ is the nerve functor. If $j < i$, $\mathfrak{C}(\Delta^n)(i, j)$ is defined to be empty. The composition law in the simplicial category $\mathfrak{C}(\Delta^n)$ is induced by the map of categories $P_{j,k} \times P_{i,k} \to P_{i,k}$ given by union of sets. The construction of $\mathfrak{C}(\Delta^n)$ is functorial with respect to simplicial maps between standard simplices. Then the functor $\mathfrak{C} : Set_\Delta \to Cat_\Delta$ is defined by $\mathfrak{C}(S) := \mathrm{colim}_{\Delta^n \to S} \mathfrak{C}(\Delta^n)$.

$\mathfrak{C}$ is defined as a colimit in the category of simplicial categories. Dugger and Spivak computed in [DS11] the mapping spaces of $\mathfrak{C}$ explicitly via necklaces. More precisely, in Proposition 4.3 of [DS11] they prove there is an isomorphism of simplicial sets

$$\underset{T \to S \in (Nec\downarrow S)_{x,y}}{\mathrm{colim}} [\mathfrak{C}(T)(\alpha_T, \omega_T)] \cong \mathfrak{C}(S)(x, y).$$

We defined $\mathfrak{C}_{\square_c}$ having this formula in mind. We do it this way, as opposed to first defining $\mathfrak{C}_{\square_c}$ on standard simplices and then extending as a left Kan extension, to emphasize that maps of necklaces give rise to maps of cubical sets with connections and the relationship of this fact with Adams' cobar construction, as we explain later on. The mapping spaces of the functor $\mathfrak{C}_{\square_c}$ are cubical sets with connections constructed by applying the Yoneda embedding to the category $P_{\mathbf{1}}(T)$ associated to a necklace T and then taking a colimit, while the mapping spaces in $\mathfrak{C}$ are simplicial sets obtained by applying the nerve functor to $P_{\mathbf{1}}(T)$ and then taking a colimit.

Recall we have a triangulation functor $|\cdot| : Set_{\square_c} \to Set_\Delta$ defined on a cubical set with connections K by $|K| := \mathrm{colim}_{\square^n_c \to K} N(\mathbf{1}^n) \cong \mathrm{colim}_{\square^n_c \to K} (\Delta^1)^{\times n}$. Define a functor $\mathfrak{T} : Cat_{\square_c} \to Cat_\Delta$ as follows. Given a category $\mathcal{K}$ enriched over $Set_{\square_c}$ define $\mathfrak{T}(\mathcal{K})$ to be the simplicial category whose objects are the objects of $\mathcal{K}$ and whose mapping spaces are given by $|\mathcal{K}(x, y)|$ for any objects x and y in $\mathcal{K}$. We have a composition law on $\mathfrak{T}(\mathcal{K})$ induced by applying the functor $|\cdot|$ to the composition law in $\mathcal{K}$ and using the fact that for cubical sets with connections K and K' we have a natural isomorphism $|K \otimes K'| \cong |K| \times |K'|$.

Proposition 5.3. *The functor $\mathfrak{C} : Set_\Delta \to Cat_\Delta$ is naturally isomorphic to the composition of functors*

$$Set_\Delta \xrightarrow{\mathfrak{C}_{\square_c}} Cat_{\square_c} \xrightarrow{\mathfrak{T}} Cat_\Delta.$$

Proof. Let $Y(\square_c) \downarrow \square_c^N$ be the category whose objects are morphisms $\square_c^n \to \square_c^N$ of cubical sets with connections and whose morphisms are given by the corresponding commutative triangles. Note $|\square_c^N|$ is the colimit in simplicial sets of the functor $Y(\square_c) \downarrow \square_c^N \to Set_\Delta$ that sends an object $(\square_c^n \to \square_c^N)$ to $N(\mathbf{1}^n) \cong (\Delta^1)^{\times n}$ and a morphism in $Y(\square_c) \downarrow \square_c^N$ to the corresponding induced morphism between nerves. The identity morphism $\square_c^N \to \square_c^N$ is a terminal object in $Y(\square_c) \downarrow \square_c^N$. Therefore, $|\square_c^N| = \mathrm{colim}_{\square_c^n \to \square_c^N} N(\mathbf{1}^n)$ is given by the value of the functor on the identity morphism $\square_c^N \to \square_c^N$, so $|\square_c^N| = N(\mathbf{1}^N)$. $\square$

Let S be a simplicial set. The objects of the simplicial categories $\mathfrak{T}(\mathfrak{C}_{\square_c}(S))$ and $\mathfrak{C}(S)$ are the same, i.e. the elements of S_0. Since the triangulation functor $|\cdot|$ commutes with colimits we have the following natural isomorphisms

$$(\mathfrak{T}(\mathfrak{C}_{\square_c}(S)))(x,y) \cong \underset{T \to S \in (Nec \downarrow S)_{x,y}}{\mathrm{colim}} |C_{\square_c}(T)| \cong \underset{T \to S \in (Nec \downarrow S)_{x,y}}{\mathrm{colim}} N(\mathbf{1}^{|V_T - J_T|}).$$

Moreover, by Proposition 4.3 of [DS11] it follows that we have natural isomorphisms

$$\underset{T \to S \in (Nec \downarrow S)_{x,y}}{\mathrm{colim}} N(\mathbf{1}^{|V_T - J_T|}) \cong \underset{T \to S \in (Nec \downarrow S)_{x,y}}{\mathrm{colim}} [\mathfrak{C}(T)(\alpha, \omega)] \cong \mathfrak{C}(S)(x,y).$$

Hence, we have an isomorphism of simplicial categories $\mathfrak{T}(\mathfrak{C}_{\square_c}(S)) \cong \mathfrak{C}(S)$ which is functorial on S. It follows that $\mathfrak{T} \circ \mathfrak{C}_{\square_c}$ and $\mathfrak{C}$ are naturally isomorphic functors. $\square$

6. The left adjoint of the DG nerve functor

In [Lur11] Lurie defines a functor $N_{dg} : dgCat_k \to Set_\Delta$, called the *dg-nerve*, which is weakly equivalent to the left adjoint of the composite functor

$$Set_\Delta \xrightarrow{\mathfrak{C}} Cat_\Delta \xrightarrow{\mathfrak{Q}_\Delta} dgCat_k$$

where $\mathfrak{Q}_\Delta$ is the functor obtained by applying the normalized Moore chains (over k) functor Q_Δ on the mapping spaces. In this section we prove that the composite functor

$$\Lambda : Set_\Delta \xrightarrow{\mathfrak{C}_{\square_c}} Cat_{\square_c} \xrightarrow{\mathfrak{Q}_{\square^c}} dgCat_k,$$

where $\mathfrak{Q}_{\square^c}$ is the functor obtained by applying the normalized Moore chains functor $Q_{\square_c}$ on the mapping spaces, is left adjoint to N_{dg}.

We begin by recalling Lurie's definition of N_{dg}. Let $\mathcal{C}$ be a dg-category. For each $n \geq 0$, define $N_{dg}(\mathcal{C})_n$ to be the set of all ordered pairs of sets $(\{X_i\}_{0 \leq i \leq n}, \{f_I\})$, such that:

1. $X_0, X_1, ..., X_n$ are objects of the dg-category $\mathcal{C}$
2. I is a subset $I = \{i_- < i_m < i_{m-1} < ... < i_1 < i_+\} \subseteq [n]$ with $m \geq 0$ and f_I is an element of $\mathcal{C}(X_{i_-}, X_{i_+})_m$ satisfying

$$df_I = \sum_{1 \leq j \leq m} (-1)^j (f_{I - \{i_j\}} - f_{i_j < ... < i_1 < i_+} \circ f_{i_- < i_m < ... < i_j}).$$

The structure maps in $N_{dg}(C)$ are defined as follows. If $\alpha : [m] \to [n]$ is a nondecreasing function, then the induced map $N_{dg}(\mathcal{C})_n \to N_{dg}(\mathcal{C})_m$ is given by

$$(\{X_i\}_{0 \leq 1 \leq n}, \{f_I\}) \mapsto (\{X_{\alpha(j)}\}_{0 \leq j \leq m}, \{g_J\}),$$

where $g_J = f_{\alpha(J)}$ if $\alpha|_J$ is injective, $g_J = id_{X_i}$ if $J = \{j, j'\}$ with $\alpha(j) = i = \alpha(j')$, and $g_J = 0$ otherwise.

Theorem 6.1. *The functor $\Lambda : Set_\Delta \to dgCat_k$ is left adjoint to $N_{dg} : dgCat_k \to Set_\Delta$.*

Proof. First we show that for any standard simplex Δ^n and any dg-category $\mathcal{C}$ there is bijection

$$\theta_{n,\mathcal{C}} : dgCat_k(\Lambda(\Delta^n), \mathcal{C}) \cong Set_\Delta(\Delta^n, N_{dg}(\mathcal{C}))$$

which is functorial with respect to simplicial maps of standard simplices $\Delta^n \to \Delta^m$. Note that $Set_\Delta(\Delta^n, N_{dg}(\mathcal{C})) \cong N_{dg}(\mathcal{C})_n$. Given a dg-functor $F : \Lambda(\Delta^n) \to \mathcal{C}$ we construct an n-simplex $\theta_{n,\mathcal{C}}(F) = (\{X_0, ..., X_n\}, \{f_I\})$ in $N_{dg}(\mathcal{C})_n$. The objects of $\Lambda(\Delta^n)$ are the integers $0, 1, ..., n$ so we let $X_i = F(i)$ for $i = 0, 1, ..., n$. For every subset $I = \{i_- < i_1 < ... < i_m < i_+\} \subseteq [n]$ define σ_I to be the generator of the chain complex $\Lambda(\Delta^n)(i_-, i_+) = Q_{\square_c}(\mathfrak{C}_{\square_c}(\Delta^n)(i_-, i_+))$ represented by the non-degenerate element of $(\mathfrak{C}_{\square_c}(\Delta^n)(i_-, i_+))_m$ which is the one bead sub-necklace inside Δ^n consisting of the $(m+1)$-simplex with i_- as first vertex, i_+ as last vertex, and $i_1, ..., i_m$ as non-joint vertices, in other words, σ_I is represented by the $(m+1)$-simplex inside Δ^n spanned by vertices $i_-, i_1, ..., i_m, i_+$. It follows from Remark 4.2 that

$$d\sigma_I = \sum_{j=1}^{m} (-1)^j (\partial_j^1 \sigma_I - \partial_j^0 \sigma_I) = \sum_{j=1}^{m} (-1)^j (\sigma_{I - \{i_j\}} - \sigma_{i_j < ... < i_1 < i_+} \circ \sigma_{i_- < i_m < ... < i_j}).$$

Define $f_I = F(\sigma_I) : X_{i_-} \to X_{i_+}$. Since the dg-functor F commutes with differentials at the level of mapping spaces, f_I satisfies property (2) in the definition of the dg-nerve functor. The functoriality of $\theta_{n,\mathcal{C}}$ with respect to simplicial maps between standard simplices follows from Proposition 3.2. Finally, since the functor Λ preserves colimits, $\theta_{n,\mathcal{C}}$ induces a functorial bijection

$$dgCat_k(\Lambda(S), \mathcal{C}) \cong Set_\Delta(S, N_{dg}(\mathcal{C}))$$

for any simplicial set S and dg-category $\mathcal{C}$. $\square$

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7. Models for the based loop space and free loop space

In this section we apply the functor Λ to the simplicial set $\mathrm{Sing}(X, b)$ consisting of singular simplices on a connected topological space X with vertices at a point $b \in X$. We show that $\Lambda(\mathrm{Sing}(X, b))(b, b)$ is a differential graded associative algebra (dga) weakly equivalent as a dga to the singular chain complex $S_*(\Omega_b X; k)$. Furthermore, we show that $\Lambda(\mathrm{Sing}(X, b))(b, b)$ is isomorphic to the cobar construction of the dg coassociative coalgebra of normalized chains of $\mathrm{Sing}(X, b)$, extending Adams' original result which assumed simply connectedness. This yields a model for the free loop space of X by taking the Hochschild chain complex of $\Lambda(\mathrm{Sing}(X, b))(b, b)$.

To establish a comparison between the theories of ∞-categories and simplicial categories Lurie proves the following (Proposition 2.2.4.1 [Lur09])

Proposition 7.1. *Let S be an ∞-category containing a pair of objects x and y. Then the natural map*

$$f : |Hom_S^R(x, y)|_{Q^\bullet} \to \mathfrak{C}(S)(x, y) \tag{7.1}$$

is a weak equivalence of simplicial sets.

Let us describe the constructions in the above proposition in more detail. We denote by $|-|_{Q^\bullet} : Set_\Delta \to Set_\Delta$ the realization functor associated to a cosimplicial simplsicial set $Q^\bullet$. Moreover, for any simplicial set S we have a weak equivalence $g : |S|_{Q^\bullet} \to S$ of simplicial sets, so $|S|_{Q^\bullet}$ may be thought of as a simplicial fattening of S. We now describe the cosimplicial object $Q^\bullet : \Delta \to Set_\Delta$ and the weak equivalences $g : |S|_{Q^\bullet} \to S$, and $f : |\mathrm{Hom}^R_S(x,y)|_{Q^\bullet} \to \mathfrak{C}(S)(x,y)$. First, define a cosimplicial object $J^\bullet : \Delta \to (\partial\Delta^1 \downarrow Set_\Delta)$, by letting J^n to be the quotient of the standard simplex Δ^{n+1} by collapsing the last face (i.e. the face spanned by vertices $[0,...,n]$) to a vertex. The quotient simplicial set J^n has exactly two vertices which we denote by the integers 0 and $n+1$. Note that J^n is an object in the slice category $(\partial\Delta^1 \downarrow Set_\Delta)$. A morphism $[n] \to [m]$ in Δ clearly induces a map of simplicial sets $J^n \to J^m$. We define $Q^n := \mathfrak{C}(J^n)(0, n+1)$. Moreover, one can show that the simplicial set $\mathfrak{C}(J^n)(0, n+1)$ is a quotient of the simplicial set $\mathrm{sd}(\Delta^n)$, the simplicial barycentric subdivision of Δ^n.

We have a map of simplicial sets

$$g : |S|_{Q^\bullet} \cong \mathrm{colim}_{\Delta^n \to S}\mathfrak{C}(J^n)(0, n+1) \to \mathrm{colim}_{\Delta^n \to S}\Delta^n \cong S$$

induced by the map $\tilde{l} : \mathfrak{C}(J^n)(0, n+1) \to \Delta^n$, which is in turn induced by the last vertex map $l : \mathrm{sd}(\Delta^n) \to \Delta^n$. Lurie proves g is a weak equivalence for any simplicial set S [Lur09].

The simplicial set $\mathrm{Hom}^R_S(x,y)$ is the *right mapping space* of S between x and y and is defined by letting $\mathrm{Hom}^R_S(x,y)_n$ be the set of all morphisms of simplicial sets $\varphi : J^n \to S$ such that $\varphi(0) = x$ and $\varphi(n+1) = y$, together with structure face and degeneracy maps defined to coincide with the corresponding structure maps of on S_{n+1}. Therefore, we have

$$|\mathrm{Hom}^R_S(x,y)|_{Q^\bullet} \cong \underset{\varphi: J^n \to S \in Hom^R_S(x,y)}{\mathrm{colim}} \mathfrak{C}(J^n)(0, n+1).$$

The map of simplicial sets f in Proposition 7.1 can be identified with the canonical map

$$\underset{\varphi: J^n \to S \in Hom^R_S(x,y)}{\mathrm{colim}} \mathfrak{C}(J^n)(0, n+1) \to \mathfrak{C}(S)(x,y)$$

which takes $[\varphi : J^n \to S \in Hom^R_S(x,y), \sigma \in \mathfrak{C}(J^n)(0, n+1)]$ to $\mathfrak{C}(z)(\sigma) \in \mathfrak{C}(S)(x,y)$, where $\mathfrak{C}(\varphi) : \mathfrak{C}(J^n) \to \mathfrak{C}(S)$ is the map of simplicial categories induced by $\varphi : J^n \to S$. In other words, under the isomorphism

$$\mathfrak{C}(S)(x,y) \cong \underset{(Nec\downarrow S)_{x,y}}{\mathrm{colim}} \mathfrak{C}(T)(\alpha_T, \omega_T)$$

the above map sends an equivalence class $[\varphi : J^n \to S \in Hom^R_S(x,y), \sigma \in \mathfrak{C}(J^n)(0, n+1)]$ to the equivalence class $[\varphi \circ q : \Delta^{n+1} \to S \in (Nec \downarrow S)_{x,y}, \mathfrak{C}(\varphi \circ q)(\sigma) \in \mathfrak{C}(\Delta^{n+1})(0, n+1)]$ where $q : \Delta^{n+1} \to J^n$ is the quotient map. Lurie proves that if S is an ∞-category then f is a weak equivalence [Lur09].

For any simplicial category $\mathcal{C}$ the *simplicial nerve* $N_\Delta(\mathcal{C})$ is the simplicial set whose set of n-simplices is given by

$$(N_\Delta(\mathcal{C}))_n = \mathrm{Hom}_{Cat_\Delta}(\mathfrak{C}(\Delta^n), \mathcal{C}).$$

It follows that $N_\Delta : Cat_\Delta \to Set_\Delta$ is the right adjoint of $\mathfrak{C} : Set_\Delta \to Cat_\Delta$. If $\mathcal{C}$ is a topological category, then the *topological nerve* $N_{Top}(\mathcal{C})$ is defined to be the simplicial nerve of the simplicial category $\text{Sing}(\mathcal{C})$ obtained by applying Sing to each morphism space of $\mathcal{C}$. Lurie shows that the pair of functors $(\mathfrak{C}, N_\Delta)$ defines a Quillen equivalence between model categories Set_Δ with the Joyal model structure and Cat_Δ with the Bergner model structure. In particular, for any simplicial category $\mathcal{C}$ the counit map $\mathfrak{C}(N_\Delta(\mathcal{C})) \to \mathcal{C}$ is a weak equivalence of simplicial categories.

Let X be a connected topological space and let $x, y \in X$. We analyze the simplicial category $\mathfrak{C}(\text{Sing}(X))$ in the light of the above discussion. Define the standard space of paths in X between x and y to be the set $P_{x,y}X := \{\gamma : [0,1] \to X | \gamma(0) = x, \gamma(1) = y\}$ with the compact open topology. Define the space of Moore paths in X between x and y to be $P^M_{x,y}X = \{(\gamma, r) | \gamma : [0,\infty) \to X, \gamma(0) = x, \gamma(s) = y \text{ for } r \leq s, r \in [0,\infty)\}$ topologized as a subset of $\text{Map}([0,\infty), X) \times [0,\infty)$. The space $P_{x,y}X$ is a deformation retract of $P^M_{x,y}X$. For a point $b \in X$ denote by $\Omega_b X := P_{b,b}X$ and $\Omega^M_b X := P^M_{b,b}(X)$ the standard based loop space and Moore based loop space, respectively, of X at b.

There is a weak equivalence of simplicial sets

$$\theta : Hom^R_{\text{Sing}(X)}(x,y) \to \text{Sing}(P^M_{x,y}X)$$

given as follows. A simplex $\varphi : J^n \to \text{Sing}(X) \in Hom^R_{\text{Sing}(X)}(x,y)$ corresponds to a continuous map $\sigma_\varphi : |\Delta^{n+1}| \to X$ which collapses the last face of $|\Delta^{n+1}|$ to x and sends the last vertex of $|\Delta^{n+1}|$ to y. For each point p in the last face of $|\Delta^{n+1}|$ there is a straight line segment from p to the last vertex of $|\Delta^{n+1}|$. These straight line segments give a family of disjoint paths inside $|\Delta^{n+1}|$ which start in the last face and end in the last vertex and such family is parametrized by $|\Delta^n|$. The continuous map σ_φ induces a continuous map $|\Delta^n| \to P^M_{x,y}X$ which corresponds to a simplex $\theta(\varphi) : \Delta^n \to \text{Sing}(P^M_{x,y}X)$. The map θ is clearly a weak equivalence of simplicial sets. It follows from Proposition 7.1 that we have a weak equivalence $\mathfrak{C}(\text{Sing}(X))(x,y) \simeq \text{Sing}(P^M_{x,y}X)$.

Moreover, the weak equivalences $\mathfrak{C}(\text{Sing}(X))(x,y) \simeq \text{Sing}(P^M_{x,y}X)$ of simplicial sets form part of a weak equivalence of simplicial categories. Consider the simplicial category $\text{Sing}(\mathcal{P}X)$ obtained by applying Sing to each of the morphsim spaces in the topological category $\mathcal{P}X$. The topological category $\mathcal{P}X$ has as objects the points of X and as morphism spaces the spaces of Moore paths with composition law induced by concatenation of paths.

Proposition 7.2. *Let X be a connected topological space. The simplicial categories $\mathfrak{C}(Sing(X))$ and $Sing(\mathcal{P}X)$ are weakly equivalent.*

Proof. Choose a point $b \in X$. The topological category $\mathcal{P}X$ is weakly equivalent to the topological category ΩX which has a single object b and as morphism space $\Omega X(b,b) = \Omega^M_b X$ the space of based Moore loops at b with composition law given by concatenation of loops; this follows by choosing a path from b to every point of X. Since N_{Top} is (weakly) invariant under weak equivalences of simplicial categories, we have $N_{Top}(\mathcal{P}X) \simeq N_{Top}(\Omega X)$. Moreover, the geometric realization $|N_{Top}(\Omega X)|$ is $B(\Omega^M_b X)$, the classifying space of the topological grupoid $\Omega^M_b X$, thus $|N_{Top}(\Omega X)| \simeq X$. It follows that the simplicial sets $N_{Top}(\mathcal{P}X)$ and $\text{Sing}(X)$ are weakly equivalent.

Moreover, $N_{Top}(\mathcal{P}X)$ is a Kan complex since its homotopy category is a groupoid [Joy02] and $\mathfrak{C}$ preserves weak equivalences of Kan complexes [Rie14], hence we have $\mathfrak{C}(N_{Top}(\mathcal{P}X)) \simeq \mathfrak{C}(\mathrm{Sing}(X))$. On the other hand, we have that $\mathfrak{C}(N_{Top}(\mathcal{P}X)) = \mathfrak{C}(N_\Delta(\mathrm{Sing}(\mathcal{P}X))) \simeq \mathrm{Sing}(\mathcal{P}X)$. It follows that the simplicial categories $\mathfrak{C}(\mathrm{Sing}(X))$ and $\mathrm{Sing}(\mathcal{P}X)$ are weakly equivalent. $\square$

For a point $b \in X$ we denote by $\mathrm{Sing}(X, b)$ the subsimplicial set of $\mathrm{Sing}(X)$ whose n-simplices are continuous maps $\Delta^n \to X$ that take all vertices of Δ^n to b. If X is connected $\mathrm{Sing}(X)$ and $\mathrm{Sing}(X, b)$ are weakly equivalent Kan complexes, thus we have $\mathfrak{C}(\mathrm{Sing}(X))(b, b) \simeq \mathfrak{C}(\mathrm{Sing}(X, b))(b, b)$. The following corollary follows directly from Proposition 6.2.

Corollary 7.3. *Let X be a connected topological space and $b \in X$. There is a weak equivalence of simplicial categories $\mathfrak{C}(Sing(X, b)) \simeq Sing(\Omega X)$.*

It follows from the above corollary that $Q_\Delta(\mathfrak{C}(\mathrm{Sing}(X, b))(b, b))$ is weakly equivalent as a differential graded associative algebra to $S_*(\Omega^M_b X; k)$, the singular chain complex on the space $\Omega^M_b X$ with k coefficients. We show that $Q_{\square_c}(\mathfrak{C}_{\square_c}(\mathrm{Sing}(X, b))(b, b))$ and $S_*(\Omega^M_b X; k)$ are weakly equivalent as dga's as well by applying the following

Proposition 7.4. *For any cubical set with connections K the chain complex $Q_\Delta(T(K))$ is weakly equivalent to $Q_{\square_c}(K)$, where $T : Set_{\square_c} \to Set_\Delta$ is the triangulation functor.*

Proof. This follows from an acyclic models argument applied to the two functors

$$Q_\Delta \circ T, Q_{\square_c} : Set_{\square_c} \to Ch_k.$$

Define the collection of models in $Set_{\square_c}$ to be the collection $\mathcal{M} = \{\square^0_c, \square^1_c, ...\}$ of standard cubes with connections. It is clear that both $Q_\Delta \circ T$ and $Q_{\square_c}$ are acyclic on these models. Recall a functor $F : \mathcal{C} \to Ch_k$ is *free* on $\mathcal{M}$ if there exist a collection $\{M_j\}_{j \in \mathcal{J}}$ where each M_j is an object in $\mathcal{M}$ (possibly with repetitions, possibly not including all of the objects in $\mathcal{M}$) together with elements $m_j \in F(M_j)$ such that for any object X of $\mathcal{C}$ we have that $\{F(f)(m_j) \in F(X) | j \in \mathcal{J}, (f : M_j \to X) \in \mathcal{C}(M_i, X)\}$ forms a basis for $F(X)$. Clearly $Q_{\square_c}$ is free on $\mathcal{M}$ since we can take $M_j = \square^j_c, \mathcal{J} = \{0, 1, 2, ..., \}$, and define $m_j \in Q_{\square_c}(M_j) = Q_{\square_c}(\square^j_c)$ to be the generator corresponding to the unique non-degenerate element in $(\square^j_c)_j$ (i.e. m_j is the top non-degenerate cell of $\square^j_c$). Note that the simplicial set $T(\square^j_c) \cong (\Delta^1)^{\times j}$ has $j!$ non-degenerate j-simplices $\sigma^j_1, ..., \sigma^j_{j!} \in T(\square^j_c)_j$. Hence, $Q_\Delta \circ T$ is also free on $\mathcal{M}$ since we can take $\{M^0_1, M^1_1, M^2_1, M^2_2, ..., M^j_1, ..., M^j_{j!}, M^{j+1}_1, ...\}_{j \in \mathcal{J}}$ where $M^j_k = \square^j_c, \mathcal{J} = \{0, 1, 2, ...\}$, and $m^j_k \in Q_\Delta(T(M^j_k))$ the generator corresponding to the j-simplex $\sigma^j_k \in T(\square^j_c)_j$.

We have a natural isomorphism of functors $H_0(Q_\Delta \circ T) \cong H_0(Q_{\square_c})$, in fact, for any $K \in Set_{\square_c}$ we have a natural bijection between $T(K)_0$ and K_0 and any two vertices x and y are connected by a sequence of 1-simplices in $T(K)_1$ if and only if they are connected by a sequence of 1-cubes in K_1. By the acyclic models theorem there exists natural transformations $\phi : Q_\Delta \circ T \to Q_{\square_c}$ and $\psi : Q_{\square_c} \to Q_\Delta \circ T$ such that each composition $\phi \circ \psi$ and $\psi \circ \phi$ is chain homotopic to the identity map. $\square$

Corollary 7.5. *Let X be a connected topological space and $b \in X$. The differential graded associative algebras $\Lambda(Sing(X, b))(b, b)$ and $S_*(\Omega^M_b X; k)$ are weakly equivalent.*

Proof. By definition $\Lambda(\mathrm{Sing}(X,b))(b,b) = Q_{\square_c}(\mathfrak{C}_{\square_c}(\mathrm{Sing}(X,b))(b,b))$. By Proposition 6.4 we have

$$Q_{\square_c}(\mathfrak{C}_{\square_c}(\mathrm{Sing}(X,b))(b,b)) \simeq Q_\Delta(T(\mathfrak{C}_{\square_c}(\mathrm{Sing}(X,b))(b,b))).$$

This is a weak equivalence of dga's since compositions are preserved under the triangulation functor. By Proposition 4.3, we have an isomorphism

$$Q_\Delta(T(\mathfrak{C}_{\square_c}(\mathrm{Sing}(X,b))(b,b))) \cong Q_\Delta(\mathfrak{C}(\mathrm{Sing}(X,b))(b,b)).$$

Finally, by Corollary 6.3, we have

$$Q_\Delta(\mathfrak{C}(\mathrm{Sing}(X,b))(b,b)) \simeq S_*(\Omega_b MX; k).$$

. □

Remark 7.6. A generator ξ of degree n in the chain complex $\Lambda(\mathrm{Sing}(X,b))(b,b)$ is an equivalence class which may be represented by a non-degenerate n-cell σ in the cubical set with connections $\mathfrak{C}_{\square_c}(\mathrm{Sing}(X,b))(b,b)$. The non-degenerate n-cell σ is itself an equivalence class $[r : \Delta^{n_1} \vee ... \vee \Delta^{n_k} \to \mathrm{Sing}(X,b) \in (Nec \downarrow \mathrm{Sing}(X,b))_{b,b}]$, where $n_1 + ... + n_k - k = n$ where r has the property such that there is no $u : \Delta^{m_1} \vee ... \vee \Delta^{m_l} \to \mathrm{Sing}(X,b) \in (Nec \downarrow \mathrm{Sing}(X,b))_{b,b}$ with $m_1 + ... + m_l - l < n$ together with a map of necklaces $f : \Delta^{n_1} \vee ... \vee \Delta^{n_k} \to \Delta^{m_1} \vee ... \vee \Delta^{m_l}$ satisfying $r = u \circ f$. Moreover, any $(s : \Delta^{n_1} \vee ... \vee \Delta^{n_i} \vee \Delta^1 \vee \Delta^{n_{i+1}} \vee ... \vee \Delta^{n_k} \to \mathrm{Sing}(X,b))$ satisfying $r \circ \pi = s$, where $\pi : \Delta^{n_1} \vee ... \vee \Delta^{n_i} \vee \Delta^1 \vee \Delta^{n_{i+1}} \vee ... \vee \Delta^{n_k} \to \Delta^{n_1} \vee ... \vee \Delta^{n_k}$ is the map of simplicial sets which collapses the $(i+1)$-th bead in the domain necklace to a point, also represents the equivalence class σ. This follows essentially from Proposition 3.2 (3).

Given a differential graded coassociative coalgebra (C, ∂, Δ) over a commutative ring k such that $C_0 = k$ and C is k-free on each degree, the *cobar construction* of C is the differential graded associative algebra $(\Omega C, D)$ defined as follows. Consider the graded k-module sC where s is the shift by -1, i.e. $(sC)_i = C_{i+1}$ for $i \geq 0$ and $(sC)_i = 0$ for $i < 0$. Let $\Delta = \mathrm{Id} \otimes 1 + 1 \otimes \mathrm{Id} + \Delta'$ and for any $c \in C$ write $\Delta'(c) = \sum c' \otimes c''$. The underlying algebra of the cobar construction is $\Omega C = TsC = k \oplus sC \oplus (sC \otimes sC) \oplus (sC \otimes sC \otimes sC) \oplus ...$ and the differential D is defined by extending $D(sc) = -s\partial c + \sum (-1)^{\deg c'} sc' \otimes sc''$ as a derivation to all of ΩC.

Let $N_*^b(X;k) := Q_\Delta(\mathrm{Sing}(X,b))$, so $(N_*^b(X;k), \partial)$ is the normalized chain complex over k of singular chains on X with vertices on b. The Alexander-Whitney diagonal map Δ induces a dg coassociative coalgebra structure on $N_*^b(X;k)$. Denote by $\Omega(N_*^b(X;k), \partial, \Delta)$ the cobar construction of the dg coassociative coalgebra $(N_*^b(X;k), \partial, \Delta)$.

Theorem 7.7. *Let X be a connected topological space and $b \in X$. There is an isomorphism of differential graded algebras $\Lambda(\mathit{Sing}(X,b))(b,b) \cong \Omega(N_*^b(X;k), \partial, \Delta)$.*

Proof. Let $S_*(X;k)$ be the (unnormalized) singular chain complex of X and let $S_*^b(X;k)$ be the (unnormalized) singular chain complex of X with vertices at b. The boundary map $\partial : S_n(X;k) \to S_{n-1}(X;k)$ and the Alexander-Whitney diagonal $\Delta : S_n(X;k) \to \bigoplus_{p+q=n} S_p(X;k) \otimes S_q(X;k)$ induce a differential graded coassociative coalgebra on $S_*^b(X;k)$. Also note that the maps $\partial : S_n^b(X,k) \to S_{n-1}^b(X;k)$ and $\Delta : S_n^b(X;k) \to \bigoplus_{p+q=n} S_p^b(X;k) \otimes S_q^b(X;k)$ can be written as alternating sums $\partial = \sum_{i=0}^{n}(-1)^i \partial_i$ and $\Delta = \sum_{i=0}^{n}(-1)^i\Delta_i$ as usual. The truncated maps $\partial' = \sum_{i=1}^{n-1}(-1)^i\partial_i$ and $\Delta' = \sum_{i=1}^{n-1}(-1)^i\Delta_i$ also define a differential graded coassociative coalgebra structure on $S^b_*(X;k)$ (in fact, it is a contractible dg coassociative coalgebra). We may form a dga by taking the cobar construction $\Omega(S^b_*(X;k),\partial',\Delta')$. First, we show $\Lambda(\mathrm{Sing}(X,b))(b,b) = Q_{\square_c}(\mathfrak{C}_{\square_c}(\mathrm{Sing}(X,b))(b,b)) \cong \Omega(S^b_*(X;k),\partial',\Delta')/\sim$ for some equivalence relation $\sim$ and then we show $\Omega(S^b_*(X;k),\partial',\Delta')/\sim \;\cong \Omega(N^b_*(X;k),\partial,\Delta)$.

Recall $\Omega(S^b_*(X;k),\partial',\Delta')$ is a dga which has as underlying complex the tensor algebra $TsS^b_{*>0}(X;k)$, where s is the functor that shifts degree by -1, together with differential $D'_\Omega = \partial' + \Delta'$ extended as a derivation to all of $TsS^b_{*>0}(X;k)$. We denote a monomial $s\sigma_1\otimes ...\otimes s\sigma_k \in TsS^b_{*>0}(X;k)$ by $[\sigma_1|...|\sigma_k]$. Let $s_0(b) \in S^b_1(X,k)$ be the degenerate 1-simplex at b. We take a quotient of $TsS^b_{*>0}(X;k)$ by the equivalence relation generated by

$$[\sigma_1|...|\sigma_k] \sim [\sigma_1|...|\sigma_{i-1}|\sigma_{i+1}|...|\sigma_k]$$

if for some $1 \leq i \leq k$ we have $\sigma_1 = s_0(b)$ (in particular $[\sigma_1] \sim 1_k$ if $\sigma_1 = s_0(b)$); and

$$[\sigma_1|...|\sigma_k] \sim 0$$

if $\sigma_i \in S^b_{n_i}(X;k)$ is a degenerate simplex and $n_i > 1$ for some $1 \leq i \leq k$. The first relation is corresponds to the identification in the colimit defining $\mathfrak{C}_{\square_c}(\mathrm{Sing}(X,b))(b,b)$ as described in Remark 6.6; the second relation corresponds to modding out by degenerate chains in the definition of the chain complex of normalized Moore chains $Q_{\square_c}(\mathfrak{C}_{\square_c}(\mathrm{Sing}(X,b))(b,b))$. The algebra structure of $TsS^b_{*>0}(X;k)$ passes to $(TsS^b_{*>0}(X;k))/\sim$, so does the differential D'_Ω. It is clear by construction that the dga's $Q_{\square_c}(\mathfrak{C}_{\square_c}(\mathrm{Sing}(X,b))(b,b))$ and $\Omega(S^b_*(X;k),\partial',\Delta')/\sim$ are isomorphic since necklaces in $\mathrm{Sing}(X,b)$ correspond to monomials of generators in $S^b_*(X;k)$

We define an isomorphism of dga's

$$\tilde{\varphi} : \Omega(S^b_*(X;k),\partial',\Delta')/\sim \;\to \Omega(N^b_*(X;k),\partial,\Delta).$$

Given $\sigma \in S^b_*(X;k)$ denote by $\overline{\sigma}$ the equivalence class of σ in $N^b_*(X;k)$. First define $\varphi[\sigma] = [\overline{\sigma}]$ if $\deg\sigma > 1$, $\varphi[\sigma] = \overline{\sigma} + 1$ if $\deg\sigma = 1$, and $\varphi(1) = 1$. Extend φ as an algebra map to obtain a map $\varphi : \Omega(S^b_*(X;k),\partial',\Delta') \to \Omega(N^b_*(X;k),\partial,\Delta)$. It follows by a short computation that the map φ is a chain map. Moreover, it induces a map of dga's $\tilde{\varphi} : \Omega(S^b_*(X;k),\partial',\Delta')/\sim \;\to \Omega(N^b_*(X;k),\partial,\Delta)$. The map $\tilde{\varphi}$ is an isomorphism of dga's, in fact, the inverse map $\psi : \Omega(N^b_*(X;k),\partial,\Delta) \to \Omega(S^b_*(X;k),\partial',\Delta')/\sim$ is given by defining $\psi[\overline{\sigma}] = \big[[\sigma]\big]$ if $\deg\sigma > 1$, $\psi[\overline{\sigma}] = \big[[\sigma]\big] - 1$ if $\deg\sigma = 1$, and $\psi(1) = 1$; then extending ψ as an algebra map. Here $\big[[\sigma]\big]$ denotes the equivalence class of $[\sigma]$ in the quotient $\Omega(S^b_*(X;k),\partial',\Delta')/\sim$ and it is clear that ψ is a well defined map of dga's inverse to $\tilde{\varphi}$.

$\square$

Remark 7.8. The normalized chains construction, when equipped with the Alexander-Whitney diagonal map, can be thought of as a functor $Q_\Delta : Set_\Delta \to dgCoalg_k$, where $dgCoalg_k$ is the category of differential graded coassociative connected coalgebras over k. Let Set^0_Δ be the full subcategory of Set_Δ whose objects are simplicial sets with only one vertex. Proposition 6.4 together with a similar proof to the one above yields the following more general result: The functor $Q_\Delta : Set^0_\Delta \to dgCoalg_k$ sends *categorical equivalences* to *weak equivalences of dg coalgebras*. A categorical equivalence (also called a Joyal equivalence) is a map of simplicial sets $f : S \to S'$ inducing a weak equivalence of simplicial categories $\mathfrak{C}(f) : \mathfrak{C}(S) \to \mathfrak{C}(S')$; a weak equivalence of dg coalgebras is a map of dg coalgebras $g : C \to C'$ inducing a quasi-isomorphism between cobar constructions $\Omega g : \Omega C \to \Omega C'$.

It follows that the cobar construction of the dg coassociative coalgebra of normalized chains on a connected (possibly non-simply connected) space X gives a dga model for the based loop space as shown in the following

Corollary 7.9. *For any connected topological space X with $b \in X$, the differential graded associative algebras $\Omega(N^b_*(X;k), \partial, \Delta)$ and $S_*(\Omega^M_b X; k)$ are weakly equivalent.*

Proof. This follows directly from Corollary 6.5 and Theorem 6.7. □

Remark 7.10. Another explicit and smaller description of a model for the based loop space in the case of a simplicial complex may be given by applying the methods discussed in this section to the Kan fibrant replacement functor. Let K be a simplicial complex with an ordering of its vertices and let v be a vertex of K. Let fK be the simplicial set generated by adding degeneracies freely to K. Consider the Kan fibrant replacement $\mathrm{Ex}^\infty(fK)$ of fK. $\mathrm{Ex}^\infty(fK)$ is a Kan complex weakly equivalent to fK, so it follows that the Kan complexes $\mathrm{Ex}^\infty(fK)$ and $\mathrm{Sing}(|K|)$ are weakly equivalent. Thus $\mathfrak{C}(\mathrm{Ex}^\infty(fK))$, $\mathfrak{C}(\mathrm{Sing}(|K|))$, and $\mathrm{Sing}(\mathcal{P}|K|)$ are weakly equivalent simplicial categories. Therefore $\Lambda(\mathrm{Ex}^\infty(fK))(v,v)$ is a dga model for the based loop space of $|K|$ at v. This remark explains an example of Kontsevich outlined in [Kon09].

Corollary 7.11. *For any connected topological space X with $b \in X$, the Hochschild chain complex of the dga $\Omega(N^b_*(X;k), \partial, \Delta)$ is weakly equivalent to the chain complex $S_*(LX;k)$ of singular chains on LX, the free loop space of X.*

Proof. This is a direct consequence of a theorem of Goodwillie [Goo85], Corollary 6.8, and the invariance of Hochschild chains under weak equivalences of dga's. □

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